INTRODUCTORY RELATIVITY

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To Olive

Pure Mathematical Physics

CONTENTS

PREFACE

This textbook is designed for first and second year undergraduates in Physics, Engineering and Mathematics. Generally, only simple algebra and calculus are used, at a level the average student should understand. The text is developed from the author's, *An Introduction to the Theory of Relativity*, though the opportunity has been taken to add additional material and to change the emphasis, in several places. The author has found that many students are interested enough in relativity to want to read around the subject. Consequently, the present text contains more than is generally given in a first lecture course on relativity, so that the text can be used to extend lecture courses.

The present text follows the historical approach, in which special relativity is presented as a natural consequence of the extension of the principle of relativity to the laws of classical optics and electromagnetism. However, most of the historical work on the ether theories has either been omitted or relegated to Appendix 1. The author has found that, with elementary students, it is better to proceed as directly as possible from Newtonian mechanics to the theory of special relativity. With such students it is hardly worth while cultivating the confusing ether points of view in detail, merely to replace them immediately with the relativistic point of view. *If* they wish, they can return to study the ether theories in detail later, as part of the history of science, after cultivating the relativistic point of view.

The Lorentz transformations are developed in the conventional way in Chapter 3, using the principle of relativity and the principle of the constancy of the speed of light. An alternative approach, based on the same axioms, but using radar methods to determine the positions and times of events, is given in Appendix 6. This is Professor H. Bondi's method of the *K*-calculus. The author has found that students find this method of measuring the positions and times of distant events easier to follow than using spatially distributed synchronized clocks. It is to be hoped that more and more teachers will try the approach of Appendix 6.

In Chapter 5, the Lorentz transformations are used as a heuristic aid to develop the laws of relativistic mechanics. There is now, however, independent experimental evidence in favour of relativistic mechanics, which can now be developed from experiments, independently of the Lorentz transformations. The theory of special

relativity can now be presented as a natural consequence of the extension of the principle of relativity to these new experimental laws of high-speed mechanics. An account of this method is given in Appendix 4. The Lorentz transformations can be developed from the principle of the constancy of the limiting speed of particles, developed in Section 5.4.2 and Appendix 3(b). Some readers may find it more convincing to approach special relativity from the viewpoint of experiments on high-speed particles, since in this case deviations from Newtonian mechanics are large. When approached via optics and electromagnetism, most of the experimental evidence in favour of special relativity is the absence of some small second order effects, which should have been present if the Galilean transformations were correct (cf. Appendix 1). The approach to special relativity via mechanics gives the reader an alternative point of view, based on experiments with particles, rather than waves.

In Chapter 6, the two methods of representing the Lorentz transformations geometrically are discussed. Minkowski's method of using real variables is a natural extension of the K-calculus methods of Appendix 6. Four-vector methods are used to redevelop special relativity in Chapter 6.

Not all students want a comprehensive discussion of relativistic electromagnetism in their first course in relativity, so that only a very brief insight into relativistic electromagnetism is given in Chapter 7, of the present text. More advanced readers can go on to *Introduction to the Theory of Relativity* or to the author's forthcoming book *Electromagnetism via Relativity, an Alternative Approach to Maxwell's Equations* for comprehensive discussions of conventional relativistic electromagnetism. The latter book also includes the development and interpretation of Maxwell's equations from Coulomb's law, using the force transformations of special relativity.

A full account of the Clock Paradox is given in Chapter 8. In Chapter 9, a brief account is given of some of the features of the theory of general relativity based on the Principle of Equivalence. This Chapter illustrates some of the limitations of the special (or restricted) theory. A large number of problems is given at the end of most chapters.

The author would like to thank Professor G. K. T. Conn for his encouragement throughout the preparation of the manuscript.

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HISTORICAL INTRODUCTION I—NEWTONIAN MECHANICS

1.1. INTRODUCTION

The enormous success of Newtonian mechanics in interpreting many of the phenomena familiar to them in their daily lives gives students the impression that it is infallible and correct in every detail. It is surprising how many students are hazy about the fundamentals of Newtonian mechanics, even about such things as the definition of inertial mass and force. At school level the understanding of physics is largely based on familiarity with the equations of physics. When problems are set in mechanics, if they can use the appropriate equation to obtain the correct solution to the problem, students feel that they understand the subject. In this chapter some features of Newtonian mechanics will be discussed a little more critically than is normally done at school level, in preparation for the changes in interpretation necessary when one comes to discuss the theory of special relativity. A few features of scientific theories in general will be pointed out within the context of Newtonian mechanics. This may help to make it a little easier for those readers meeting relativity for the first time to accept that Newtonian mechanics is not infallible, and that, when the velocities of the particles are comparable with the velocity of light, then Newtonian mechanics must be replaced by an entirely new theory.

1.2. THE STANDARDS OF LENGTH, TIME AND MASS

The feature which marked the rise of physical science in the sixteenth and seventeenth centuries was that scientists began to make quantitative observations and to carry out experiments under controlled conditions. In order to make quantitative measurements, standards have to be chosen. This book begins with a review of the primary standards of the so-called fundamental units of length, time and mass.

The international standard of length is the metre. Until recently it was defined as the distance between two defining marks engraved on a platinum–iridium bar, when the temperature of the bar was

that of melting ice. The standard metre was kept at the International Bureau of Weights and Measures at Sèvres near Paris. Accurate copies of this standard metre were made and these secondary standards were then used to calibrate other secondary devices such as rulers, which were then used for measuring lengths. The standard metre was calibrated in terms of the wavelength of the red cadmium line and other spectral lines. For example, using a Fabry and Perot interferometer, it was shown by Benoit, Fabry and Perot in 1913 that there were 1,553,164·13 wavelengths of the red cadmium line in a metre of dry air at 15°C and 760 mm pressure. The wavelength of the cadmium line under these conditions was 6438·4696 Å. This wavelength was used for many years as a secondary standard of length and could have been used to calibrate a new standard metre to an accuracy of at least one part in a million. At the Eleventh General Conference on Weights and Measures meeting in Paris in October 1960 it was decided to replace the old definition of the metre in terms of the platinum–iridium prototype by the statement that the metre *is* 1,650,763·73 wavelengths of the orange–red line of krypton–86. The new metre agrees with the old to within about one part in 10^7. If one were in a rocket, moving with uniform velocity relative to the earth, one could still use the new definition of the metre as the unit of length, and one could build an optical apparatus from materials in the rocket and use it to calibrate secondary standards such as rulers, which were at rest relative to the rocket.

In order to measure time one must choose a repetitive process and assume that it recurs at constant time intervals. Any repetitive process could be chosen as a primary standard; the one actually chosen was the mean solar day, which was defined in terms of the average time taken by the earth to make one complete revolution on its axis with respect to the sun. On account of the orbital motion of the earth around the sun, the solar day varies during the year; hence the average value was taken as the standard. Astronomers prefer to take the period of rotation of the earth relative to the 'fixed' stars, and they use sidereal time. In the laboratory, clocks such as an oscillating pendulum can be calibrated in terms of the mean solar day. The frequency of electrical signals can be measured very accurately and quartz crystal clocks can be used as accurate secondary standards. The frequencies associated with some atomic and nuclear processes have also been used as secondary standards of frequency, for example in ammonia vapour and caesium clocks. There may be secular changes in the angular velocity of the earth's rotation so that the mean solar day may vary from year to year. Pending a possible re-definition of the second in terms of an atomic or

molecular frequency, the Eleventh General Conference on Weights and Measures decided to define *the* second as 1/31,556,925·9747 of the tropical year 1900. This fixes the unit of time. If one were in a rocket moving relative to the earth one could set up a secondary atomic clock and calibrate other secondary clocks at rest in the rocket in terms of the frequency of an atomic process, the atomic frequency having previously been determined in terms of the solar year of 1900 using apparatus at rest on the earth.

The international primary standard of mass is the kilogram; this is defined as the mass of a cylinder of platinum–iridium kept at Sèvres. The comparison of the gravitational masses of other bodies with the standard can be carried out by weighing. The determination of inertial mass involves the application of the theory of mechanics and a discussion of this is deferred until after the consideration of Newton's laws of motion. In principle, the rest mass of the proton could be used as a secondary standard of mass suitable for calibrating masses at rest on a rocket moving with uniform velocity relative to the earth.

Other mechanical quantities are expressed in terms of mass, length and time. For example, the velocity of a body is defined as the distance the body moves in unit time. The velocity of light *in vacuo* can be measured accurately, and it has been suggested that one could adopt the velocity of light *in vacuo* as a primary standard, assigning it an arbitrary value. A time interval could then be measured in terms of the distance travelled by light *in vacuo* in that interval. Alternatively, one could measure a distance in terms of the time light or radio signals take to travel that distance. This is the method used in determining distances by radar methods. (Cf. Section 3.9 and Appendix 6.)

Physical measurements on bodies are carried out by comparison with the primary standards (or with secondary standards which have themselves been calibrated by comparison with the primary or other secondary standards). For example, the length of a stationary body can be determined in practice by comparing the positions of the ends of the body relative to a secondary standard such as a ruler, whose length has been subdivided into equal intervals. In this way an *experimental* determination of the length of the stationary body is made, and the physicist is satisfied with a statement that the measured length of a body is a number between, say, 13·3 and 13·4 cm. Generally the physicist does not stop to worry about terms such as the correct or absolute length of a body, though, of course, he is always striving to improve the accuracy of his experimental measurements.

It is common practice to use symbols to denote the magnitudes of physical quantities relative to the primary standards. This enables the laws of physics to be written in a mathematical form. The mathematical expressions of the laws of mechanics do not depend on the actual sizes of the arbitrary units of mass, length and time. The sizes of the fundamental units chosen affect only the values of the numerical constants in the mathematical equations expressing the laws of physics.

1.3. A CRITIQUE OF NEWTONIAN MECHANICS

Prior to the time of Galileo, the prevailing views on motion were derived largely from Aristotle, and when they were not derived from pure reason without recourse to experiments, they were generally based on *qualitative* observations only. In the sixteenth and seventeenth centuries under the influence of people such as Copernicus, Tycho Brahe, Kepler, Galileo, etc., a new outlook arose based on quantitative observations and systematic experimentation under controlled conditions. From a limited number of observations laws were postulated. For example, from a study of the time taken by metal balls to roll down an inclined plane, Galileo postulated the law of falling bodies. At this stage in the development of science, physical laws were generally derived directly by 'induction' from a limited number of quantitative observations. The early investigations on mechanics culminated in Newton's laws of motion and Newton's theory of universal gravitation. A typical statement of Newton's laws of motion is as follows:

(*a*) All bodies continue in their state of rest or of uniform motion in a straight line unless they are compelled to change that state by external forces.

(*b*) The rate of change of momentum is proportional to the impressed force and takes place in the direction in which the force is acting.

(*c*) Action and reaction are equal and opposite.

If in a reference system a body not under the influence of any forces (i.e. a body far removed from all other bodies capable of exerting forces), moves in a straight line with constant speed, then Newton's first law is valid in this reference system. Such a reference system is called an inertial frame. The properties of inertial frames are elaborated in Section 1.6.

According to Newton's law of universal gravitation, every particle of matter in the universe attracts every other particle with a force proportional to the product of the masses of the particles, inversely

proportional to the square of the distance between them, and directed along the line joining the particles. Thus, if the gravitational masses of two 'point' particles are m_1 and m_2 respectively, and if r is their distance apart, then

$$f = G\frac{m_1 m_2}{r^2} \tag{1.1}$$

where G is the gravitational constant.

The object of a theory is to correlate laws so that a theory is more comprehensive than a single law. Newton was familiar with the law of falling bodies and with Kepler's laws of planetary motion. Newton suggested that, if it is assumed that every pair of particles in the universe attract each other with a force given by eqn (1.1), then on the basis of this assumption, plus Newton's laws of motion, the other individual laws can be derived and many other phenomena, such as the motion of the moon around the earth, can be interpreted. In addition to interpreting known laws, new theories are used to make new predictions, which should subsequently be tested by experiment. The degree of acceptance of a new theory depends largely on how well these new predictions agree with the experimental results.

In a theory it is postulated that nature behaves in a particular way. For this reason a theory is sometimes described as a model of nature; it is not necessarily a mechanical model but may be a functional relation such as eqn (1.1). A theory should enable one to predict the course of an experiment from given initial conditions. Mathematical reasoning is used in applying a theory to a particular case. The conclusions of the theory obtained in this way should be in agreement with the experimental results. If a theory were perfectly correct, then there would be a one to one correspondence between the predictions of the theory and the course of nature. When applying a theory one idealizes the system and considers only those quantities which produce effects of the order of magnitude of the accuracy required, for example, when calculating the acceleration of a ball near the surface of the earth, in the interests of simplicity, one would neglect the gravitational attraction of a distant star, though, in principle, it would always be present.

If a theory is to describe what happens in practice, then the quantities appearing in the mathematical expressions of the theory must relate directly or indirectly to quantities which can be measured in practice. The meaning of the quantities used in the statement of Newton's laws of motion will now be discussed. The measurement of length and time relative to arbitrary standards

was discussed in Section 1.2. The velocity $\mathbf{u}$ and the acceleration $\mathbf{a}$ of a particle can be defined and measured in terms of length and time. Using Newton's second law, the force $\mathbf{f}$ acting on a particle can be defined as the rate of change of the momentum of the particle, where the momentum of the particle is defined as the product of its inertial mass m and its velocity $\mathbf{u}$, that is

$$\mathbf{f} = \frac{\mathrm{d}}{\mathrm{d}t}(m\mathbf{u}) \tag{1.2}$$

Provided the inertial mass of the particle is independent of its velocity, eqn (1.2) can be written as

$$\mathbf{f} = m\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = m\mathbf{a} \tag{1.3}$$

where $\mathbf{a}$ is the acceleration of the particle. Hence force can be measured experimentally in terms of the acceleration the force produces on a body of known inertial mass; but before force can be measured in this way one must know the inertial mass of the body. If equal forces acted on two bodies of inertial masses m_1 and m_2, then, according to eqn (1.3), the ratio of the accelerations of the bodies would be in the inverse ratio of their inertial masses. Newton's third law states that action and reaction are equal and opposite. Hence, if the two bodies collided, then the instantaneous values of the forces acting on the two bodies should be equal and opposite. Let a body of inertial mass m_1 moving with velocity u_1 collide head on with a body of inertial mass m_2, which is at rest before the collision. Let both the bodies m_1 and m_2 move in the same direction after the collision with velocities u_1' and u_2' respectively. If $\mathbf{f}_1$ and $\mathbf{f}_2$ are the instantaneous values of the forces acting on m_1 and m_2 at any instant during the collision, then according to Newton's third law

$$\mathbf{f}_1 = -\mathbf{f}_2$$

Integrating over the time of the collision, one obtains

$$\int \mathbf{f}_1\,\mathrm{d}t = -\int \mathbf{f}_2\,\mathrm{d}t$$

that is

$$\int m_1\frac{\mathrm{d}u_1}{\mathrm{d}t}\,\mathrm{d}t = -\int m_2\frac{\mathrm{d}u_2}{\mathrm{d}t}\,\mathrm{d}t$$

Hence,

$$\int_{u_1}^{u_1'} m_1\,\mathrm{d}u_1 = -\int_0^{u_2'} m_2\,\mathrm{d}u_2$$

$$m_1(u_1' - u_1) = -m_2u_2'$$

that is

$$m_1u_1 = m_1u_1' + m_2u_2' \tag{1.4}$$

This is the law of conservation of momentum. It is important to realize that it follows automatically from Newton's laws of motion. From eqn (1.4)

$$\frac{m_1}{m_2} = \frac{u_2'}{(u_1 - u_1')} \tag{1.5}$$

Hence, if the instantaneous values of the velocities just before and just after collision are determined, then the inertial masses can be compared. One method of performing the experiment would be to suspend the masses as two ballistic pendulums and let them collide. If one of the masses, say m_1, is the primary standard of mass, then the inertial mass of the other particle can be determined. This mass can then be used as a secondary standard of inertial mass. The theory of Newtonian mechanics has to be used before one can say how the inertial masses of two bodies can be compared. Having defined and prescribed how the inertial masses of a body can be measured, then the force acting on a body can be measured by observing the acceleration the force produces on a body of known inertial mass. Eqn (1.3) can then be used to calculate the force. Thus the precise meanings of some quantities such as force and mass depend on the theory being used, and must be measured in practice as prescribed by that theory. If the experimental evidence forces one to replace a theory by a new theory, then these quantities must be redefined and measured as prescribed by the new theory, and these definitions and procedures of measuring may differ from those of the old theory. It will be found that this will have to be the case when one comes to replace Newtonian mechanics by relativistic mechanics. As new theories are developed the meaning of words, such as mass and force, keeps changing, and the meaning to be attached to any word must be interpreted within the context of the theory being used.

Another property of matter associated with the term mass arises within the context of Newton's law of universal gravitation [eqn (1.1)]. The gravitational mass of a body is a measure of that property of matter by virtue of which every particle of matter exerts a gravitational force of attraction on every other particle of matter. Inertial mass is associated with a completely different property of matter, namely the fact that a force, not necessarily of gravitational origin, must be exerted on a particle in order to accelerate it. In classical theory it is an extra hypothesis, which must be tested by experiment, to equate gravitational and inertial mass. For a body

of gravitational mass m_G falling near the surface of the earth the force of gravitational attraction is equal to $G\dfrac{m_G M_G}{R^2}$, where M_G is the gravitational mass of the earth and R is the radius of the earth. The acceleration produced by this force is equal to the ratio of the force to the inertial mass of the body on which it is acting. Hence,

$$(\text{acceleration}) = \frac{(\text{gravitational mass})}{(\text{inertial mass})}\frac{GM_G}{R^2}$$

If the gravitational mass of a body were not proportional to its inertial mass, the accelerations of different bodies in the earth's gravitational field would not always be the same. Experimentally it is found that this acceleration is the same for all bodies, so that the experimental results require the proportionality of gravitational and inertial mass. If the units are chosen appropriately, then the gravitational and inertial masses can be made numerically equal. The gravitational mass of a body can be determined by weighing, using an equal-arm balance. This is the most convenient way of determining masses experimentally. To use these values in eqn (1.3) involves the extra hypothesis of the equality of gravitational and inertial mass.

There are many other hypotheses implied in the theory of Newtonian mechanics and which are not emphasized when the subject is taught for the first time. For example, it is assumed in eqn (1.3) that the mass of a body is independent of its velocity; it is assumed that all regions of space and all directions in space are equivalent; it is generally assumed that Euclidean geometry can be used to calculate the relationships between geometrical quantities, and it is assumed that rigid bodies exist. It is shown in Section 1.4. that in Newtonian mechanics it is assumed that an absolute time exists. These extra hypotheses are not given *a priori*, and experiment must decide whether they are correct or not. In fact many of these ideas have had to be modified within the context of the theories of special and general relativity.

The advance of physics is one of successive approximation, giving rise to better models of nature at each stage as new theories are developed. When applying a theory one considers an isolated system. The solar system is an example of a very good approximation to an isolated system, since the effects from outside the solar system, such as the gravitational attractions of distant stars, whilst they are always present, are so small that they can generally be neglected in practice. One generally starts with the simplest possible system, for example, with the sun and one planet. This

enables the orbit of the planet to be calculated reasonably satisfactorily. In order to improve the agreement between the theory and observations, the perturbations due to the gravitational attractions of other planets are introduced. In this way extra variables are introduced into the system but the same theory is used. This procedure generally works satisfactorily, but sometimes a state is reached when the old theories cannot be adjusted in this way to account for the experimental results. For example, using Newton's theory of gravitation, the perturbations in the orbit of the planet Mercury due to the gravitational attractions of the other planets were not adequate to account for the precession of the perihelion of the orbit of the planet Mercury. This discrepancy could only be resolved by replacing Newton's theory of gravitation by a completely new theory, namely Einstein's theory of gravitation which is based on the theory of general relativity and to which Newton's theory of gravitation is only an approximation. Similarly, the theory of special relativity is not a series of amendments to Newtonian mechanics, but a complete replacement necessitating even a reinterpretation of mass, length and time. Since physics advances by a series of successive approximations, it is foolish to say at any stage that the process of successive approximation is over. Therefore, when one comes to replace Newtonian mechanics by relativistic mechanics, one cannot claim infallibility for the theory of special relativity, but must always admit that it in turn can be replaced at any time by a new theory, provided the new theory is in better agreement with the experimental results. In fact, special relativity has its limitations and in non-inertial frames must be replaced by the general theory of relativity.

As physics has become more complex, so the basic postulates of the theories have become more complex, and the basic postulates cannot always be tested directly by experiment, as was done for example by Cavendish in the case of Newton's law of gravitation. Modern theories are often suggested to interpret the experimental results and are not always based directly on 'induction' from experiments. The degree of acceptance of such a theory does not depend only on the agreement of the fundamental postulates with experiment, but rather on whether or not the predictions of the theory as a whole agree with the experimental facts. It is shown later that the axioms which Einstein chose as the starting point of the theory of special relativity were of this type, and initially the whole theory had to be tested *a posteriori*.

It will be shown that when the velocities of the particles involved are very much smaller than the velocity of light, the differences

between Newtonian mechanics and relativistic mechanics are negligible. Thus, when $v \ll c$, as is generally the case in everyday life, Newtonian mechanics can be used even though it is known that the equations of the theory of special relativity give a better model of nature. It is an unnecessary elaboration to use the equations of the theory of special relativity in these circumstances. The laws of Newtonian mechanics are strictly deterministic; for given initial conditions the predictions are always the same. This works well in practice for bodies large on the atomic scale, but for atomic particles Newtonian mechanics has to be replaced by quantum mechanics.

1.4. THE GALILEAN TRANSFORMATIONS

Newton assumed that an absolute space existed. For example, he wrote 'Absolute space, in its own nature and without regard to anything external, always remains similar and unmovable'.

It will be assumed, for purposes of discussion, that absolute space exists and that Newton's three laws of motion hold in absolute space, at least to a very good approximation. Consider a co-ordinate system fixed in absolute space, which will be denoted by Σ. It is useful to visualize the co-ordinate system Σ as having an array of imaginary rulers parallel to the x, y and z axes respectively, so that, in principle, the co-ordinates of an event can be read directly from the rulers. If Newton's first law is valid in Σ, then a particle P, not acted upon by any forces, should travel in a straight line relative to Σ. Let the direction of the axes of Σ be chosen such that the motion of P is in the xy plane. If the velocity of the particle P is $\mathbf{u}$ and it is at the origin at a time $t = 0$, then its position at a time t should be given by $x = u_x t$, $y = u_y t$, $z = 0$, and its path should make an angle $\tan^{-1}(u_y/u_x)$ with the x axis.

Now consider another co-ordinate system, denoted Σ', which is moving with uniform velocity v relative to absolute space (Σ) along the common x axis. Let the origins coincide at a time $t = t' = 0$, and let the directions of the y' and z' axes of Σ' coincide with the y and z axes of Σ at $t = 0$. The co-ordinate system Σ' will be considered as having its own series of imaginary rulers so that the co-ordinates of an event can be measured relative to Σ'. By an event is meant something which happens independently of any co-ordinate system, e.g. if the particle P collided with another particle both an observer at rest in Σ and one at rest in Σ' would agree that the collision took place, though they would attribute different space co-ordinates to the event.

Newton also assumed that absolute time existed. For example, he wrote: 'Absolute, true and mathematical time, of itself, and by its own nature, flows uniformly on, without regard to anything external'.

If there were an absolute time, then time intervals should be the same when measured in both Σ and Σ', and since the zero of time in both Σ and Σ' was chosen to be the instant when the origins coincided, then, for all times, $t = t'$ where t and t' are the times of an

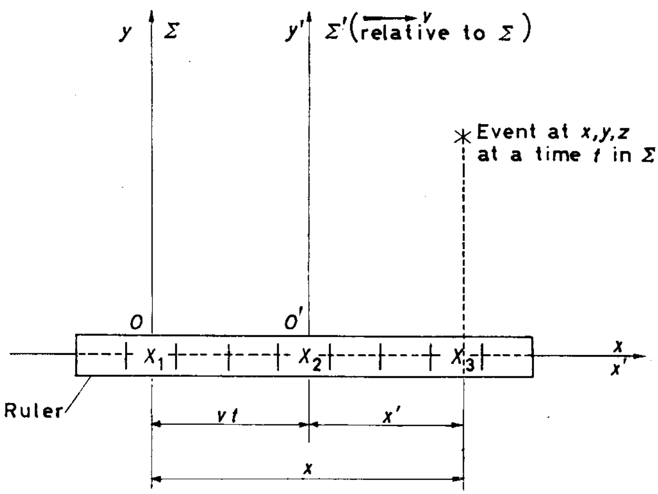

Figure 1.1. The measurement of the x and x′ co-ordinates of an event

event measured relative to Σ and Σ' respectively. The concept of absolute time was a weakness of Newtonian mechanics, since, at the time it was introduced, it was not possible to cite any experimental evidence for or against it. In the present section the concept of absolute time will be accepted as an integral part of Newtonian mechanics, even though it was introduced *a priori* without any experimental evidence.

In order to compare the x co-ordinates of an event in Σ and Σ', imagine a ruler laid out along the common x axis as shown in *Figure 1.1* (the ruler can be moving relative to both Σ and Σ'). If time were absolute, observers at rest in Σ and Σ' respectively would always agree on the time an event happened, so that both would agree that O the origin of Σ coincided with a certain position X_1 on the ruler, when O' the origin of Σ' coincided with a position X_2 on the ruler, and the x and x' co-ordinates of the event coincided with

another position X_3, as illustrated in *Figure 1.1.* Hence, if time were absolute, the observers at rest in Σ and Σ' should both agree that

$$X_1X_3 = X_1X_2 + X_2X_3$$

If it is assumed that x and x' are measured in the same units, then

$$x = OO' + x'$$

If v is the relative velocity of the co-ordinate frames, then OO' is equal to vt. Hence

$$x' = x - vt$$

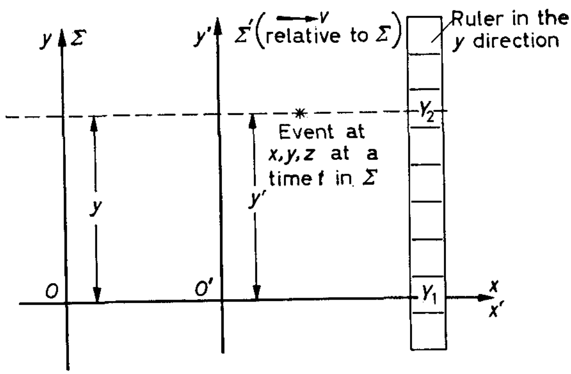

Figure 1.2. The measurement of the y and y' co-ordinates of an event

If the ruler were placed in the y direction as shown in *Figure 1.2*, then both observers should agree that their x and x' axes coincided with a position Y_1 on the ruler at the same time as the y and y' co-ordinates of the event coincided with the position Y_2, so that both observers would record the same numerical value for the y co-ordinates of the event, that is

$$y = y'$$

Similarly,

$$z = z'$$

Collecting the transformations, one has:

$$x' = x - vt \quad \text{or} \quad x = x' + vt \tag{1.6}$$

$$y' = y \tag{1.7}$$

$$z' = z \tag{1.8}$$

$$t' = t \tag{1.9}$$

These transformations are generally called the Galilean transformations.

In carrying out a measurement of the length of a moving object, the observers at rest in Σ and Σ' would observe the positions of the ends of the object on a ruler measuring both ends at the same time. If time were absolute they would agree on simultaneity. Let one end of the moving object be at a point x_1, y_1, z_1 at a time t and let the other end of the moving object be at the point x_2, y_2, z_2 in Σ at the same time t. If the co-ordinates of these events are x_1', y_1', z_1' and x_2', y_2', z_2' respectively in Σ', according to the Galilean transformations, if the events are measured at the same time in both Σ and Σ',

$$\begin{aligned}(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \\ &= (x_2' + vt - x_1' - vt)^2 + (y_2' - y_1')^2 + (z_2' - z_1')^2 \\ &= (x_2' - x_1')^2 + (y_2' - y_1')^2 + (z_2' - z_1')^2\end{aligned}$$

Thus the length of a moving object is absolute in Newtonian mechanics, that is, *it has the same numerical value in* Σ *and* Σ'. It can be seen that this result follows from the postulate of absolute time.

Class Question—How would you compare the times of two events, one on the earth and one on the moon? How would you synchronize a clock on the moon? (Comment: If you use radio signals, what assumption do you make about the speed of light in different directions of empty space? If you make any assumption at all about the speed of light in different directions, you are using a theory to help to define how to compare the times of spatially separated events.)

Now consider the particle P which is moving with uniform velocity $\mathbf{u}$ in Σ. At a time t it has co-ordinates given by

$$x = u_x t; \qquad y = u_y t; \qquad z = 0$$

Using eqns (1.6), (1.7) and (1.8), the co-ordinates of P measured in Σ' at a time $t = t'$ are

$$x' = u_x t - vt; \quad y' = u_y t; \quad z' = 0$$

The particle P, which is at the origin of Σ' at $t' = 0$ moves in the $x'y'$ plane in Σ', such that

$$\frac{x'}{y'} = \frac{x - vt}{y} = \frac{x/t - v}{y/t} = \frac{u_x - v}{u_y}$$

Since u_x, u_y and v are all constants, x'/y' is a constant, and the particle P moves in a straight line in Σ', making an angle of

$\tan^{-1}[(u_x - v)/u_y]$ with the y' axis. Thus, Newton's first law applies in Σ', if it applies in Σ.

So far only uniform motion has been considered. It will now be assumed that the particle P has an acceleration $\mathbf{a}$ relative to Σ. Differentiating eqn (1.6) with respect to time,

$$\frac{dx'}{dt} = \frac{dx}{dt} - v \quad \text{or} \quad u_x' = u_x - v \tag{1.10}$$

Similarly, differentiating eqns (1.7) and (1.8),

$$u_y' = u_y \tag{1.11}$$

and

$$u_z' = u_z \tag{1.12}$$

These are the velocity transformations. They show that the value of the velocity of a particle depends on the co-ordinate system chosen, that is, on the standard of rest chosen.

Differentiating eqns (1.10), (1.11) and (1.12) with respect to time, since v is a constant, one obtains

$$\frac{du_x'}{dt} = \frac{du_x}{dt}; \quad \frac{du_y'}{dt} = \frac{du_y}{dt}; \quad \frac{du_z'}{dt} = \frac{du_z}{dt}$$

that is

$$\mathbf{a}' = \mathbf{a} \tag{1.13}$$

In eqn (1.13) $\mathbf{a}'$ is the acceleration of the particle relative to Σ' and $\mathbf{a}$ is its acceleration relative to Σ. Provided v, the relative velocity of Σ and Σ', is constant, the accelerations are the same in Σ and Σ'. It is assumed in Newtonian mechanics that the mass of a particle is independent of its velocity, that is, mass is an invariant. Multiplying both sides of eqn (1.13) by m,

$$m\mathbf{a}' = m\mathbf{a} \tag{1.14}$$

Since the force acting on a particle can be defined as the product of the mass of the particle and the acceleration produced by the force, $m\mathbf{a}'$ is equal to the force producing the acceleration of P measured in Σ', whilst $m\mathbf{a}$ is the value of the force acting on the particle measured in Σ. Hence

$$\mathbf{f} = \mathbf{f}' \tag{1.15}$$

Thus the force on the particle, measured in terms of the acceleration it produces, should have the same value in both Σ and Σ'.

Force is not always measured in terms of the acceleration it produces. For example, if a body is suspended by an elastic string, the force of gravitational attraction is opposed by the elastic restoring force. If the body is at rest, it is not convenient to measure the elastic force in terms of an acceleration. The way the concept of force is generally used in dynamics can be illustrated by the

following example. If one wanted to calculate the acceleration of a body falling under gravity in the vicinity of the moon, one would assume that the force of gravitational attraction between the particle and the moon is given by eqn (1.1). Then using this value of force in the equation, force equals mass times acceleration, the acceleration of the body could be calculated. Thus, in Newtonian mechanics, it is sometimes assumed that the force acting on a particle is known independently of the acceleration of the particle. Using Newton's second law as a definition of force, it was concluded that if mass is absolute, force is absolute also. In Newtonian mechanics the equation $f = f'$ is extended to all types of forces; for example, it is assumed that the force due to gravitational attraction, given by eqn (1.1), is an invariant. Similarly, it is assumed that elastic forces are invariant. It has been shown that acceleration is an invariant. Hence, if mass and force are invariants, the equation force equals mass times acceleration can be used in any inertial frame, and under these conditions Newton's second law is invariant under a Galilean transformation.

At this stage the reader will have realized that there are a large number of assumptions such as absolute time, absolute mass, absolute force in Newtonian mechanics. However, within the limitation that all velocities are very much smaller than the velocity of light, Newtonian mechanics works well in practice. As an example consider a body falling off the top of the mast of a ship moving with uniform velocity. It is assumed in Newtonian mechanics that the force on the body is given by $G(mM/R^2) = mg$ relative to both the ship and the seashore. Newton's second law can be used to calculate the acceleration, and hence the motion relative to either the ship or the seashore. The co-ordinates and time at any point on the path of the particle relative to the ship and the seashore are connected by the Galilean transformations.

If two particles collide or act on each other, then, if Newton's third law is valid in Σ (absolute space), action and reaction would be equal and opposite in Σ. This means that the forces of the colliding particles on each other would always be equal and opposite when measured in Σ. According to eqn (1.15) these forces would have precisely the same numerical values when measured in Σ' as they have in Σ. If Newton's third law is valid in Σ, then it is also valid in Σ'.

It has been shown that if Newton's three laws of motion hold in absolute space (Σ), then they should hold in any co-ordinate system Σ' moving with uniform velocity relative to absolute space (Σ), if it is assumed that time, mass and force are absolute. Any one of

these co-ordinate systems could be used for recording mechanical phenomena and the results obtained could be interpreted in terms of Newton's laws of motion. If the positions and velocities of the particles in a physical system are given relative to Σ', from the forces acting on the particles, which are the same as in Σ, one can calculate the accelerations of the particles and calculate the subsequent motions of the particles relative to Σ' without any reference to absolute space. Furthermore, when Newton's laws are applied in one of the co-ordinate systems moving with uniform velocity relative to absolute space, the laws contain no term containing the velocity of the co-ordinate system relative to absolute space, but the laws take the same mathematical form in any one of the co-ordinate systems moving with uniform velocity relative to absolute space. From measurements carried out in any one of these moving co-ordinate systems, the existence of absolute space cannot be proved. Absolute space therefore loses its significance for linear motion. Thus, Newton's laws satisfy the principle of relativity when the co-ordinates and time are changed according to the Galilean transformations. Some absolute quantities still remain in Newtonian mechanics, for example, absolute time and absolute mass.

The points discussed in this section can be illustrated by considering the example of a ship going out to sea with uniform velocity on a calm day. It would be quite possible under these conditions to mark out a tennis court and to play a game of tennis on the deck of the ship, and, if there were no accelerations whatsoever, the players would not realize that the ship was moving without looking at something external to the ship. No experiment performed inside the ship, without reference to anything external to the ship, would give the velocity of the ship relative to the shore, since the laws of mechanics, determined from the experiments carried out inside the ship would not include a term containing the velocity of the ship relative to the earth. It would be perfectly satisfactory to interpret the game of tennis from the ship. An observer sitting on the seashore could also watch the game. Within the context of Newtonian mechanics the observer on the shore would agree with an observer on the ship that Newton's laws of motion were valid, at least to a good approximation. He would also agree on the dimensions of the court, the time at each stage of the game and on the score, but the two descriptions would not be completely identical. The observer on the shore would say [following from eqns (1.10), (1.11) and (1.12)] that the ball left one player's racket faster than the other's, if the ball came off both rackets at the same speed relative to the ship. The reader will probably accept it as quite natural that the speed

of the ball will be different, when the standard of rest is changed. The relations between the observer on the ship and on the shore are perfectly reciprocal. If an observer on the ship watched a game of tennis on the beach the role of the observers would be interchanged.

Newton realized that the concept of absolute space was not necessary for linear motion, e.g. he wrote:

> And thus we use, in common affairs, instead of *absolute* places and motions, *relative* ones; and that without any inconvenience. But in physical disquisitions, we should abstract from the senses. For it may be that there is no body really at rest, to which the places and motions of others can be referred.

Newton felt that absolute rotations could be observed, e.g. he wrote:

> The effects by which absolute and relative motions are distinguished from one another, are centrifugal forces, or those forces in circular motion which produce a tendency of recession from the axis.

Co-ordinate systems accelerating relative to an inertial frame will now be discussed.

1.5. NEWTON'S LAWS OF MOTION IN A CO-ORDINATE SYSTEM ACCELERATING OR ROTATING RELATIVE TO AN INERTIAL FRAME

It was pointed out in Section 1.4 that one could play a game of tennis on a ship provided the ship moved with uniform velocity. However, once the ship encountered rough seas and started to undergo sudden accelerations, one would soon realize that the laws of motion are affected by the acceleration of the reference system relative to the earth. It will now be shown how Newton's laws of motion must be modified, if one chooses as one's standard of rest a co-ordinate system accelerating or rotating relative to an inertial frame. It will again be assumed that there is an inertial frame Σ in which Newton's laws are valid, at least to a good approximation. If there are no forces acting on a particle P then it will move in a straight line in Σ. Consider a co-ordinate system (denoted A) which has a uniform linear acceleration $\mathbf{a}_r$ relative to Σ directed along the common x axis. Let the origins coincide at $t = t' = 0$ and let the co-ordinate systems be at rest relative to each other at this instant. It will be assumed that time is absolute, that is, that $t = t'$. Choose the directions of the y and z axes of Σ and the y' and z' axes of A such that the particle P moves in the xy plane in Σ with a uniform velocity $\mathbf{u}$ having components u_x, u_y, $u_z = 0$. If the particle P were at the origin of Σ at $t = 0$ then after a time t, it should be at the

point $x = u_x t$, $y = u_y t$, $z = 0$ in Σ. Its path in Σ is given by the equation $y = \dfrac{u_y}{u_x} x$.

At a time t the x co-ordinate of the origin of A relative to Σ is equal to $\frac{1}{2}a_r t^2$. If x' is the co-ordinate of the particle P relative to A at the time $t' = t$, then

$$x = x' + \tfrac{1}{2}a_r t^2$$

that is

$$x' = x - \tfrac{1}{2}a_r t^2 = u_x t - \tfrac{1}{2}a_r t^2 \tag{1.16}$$

Similarly,

$$y' = y = u_y t \tag{1.17}$$

$$z' = z = u_z t \text{ (= 0 in the present case)} \tag{1.18}$$

Substituting $t = y'/u_y$ from eqn (1.17) into eqn (1.16),

$$x' = \frac{u_x}{u_y} y' - \frac{1}{2}\frac{a_r}{u_y^2} y'^2 \tag{1.19}$$

Relative to the co-ordinate system A the particle P moves in a parabola and not in a straight line even though no real or impressed forces act on P. Hence, Newton's first law *does not hold* in the co-ordinate system A, and the particle P appears to have an acceleration $-\mathbf{a}_r$ relative to A; thus A is *not* an inertial frame. If one wanted to apply Newton's first law in the co-ordinate system A, then one would have to introduce a *fictitious* force $-m\mathbf{a}_r$ acting on the particle P of mass m in order to interpret its apparent acceleration. This fictitious force is called an inertial force; it does not produce an acceleration relative to the fixed stars. As an example of the use of a fictitious inertial force, consider a passenger on a moving train. If the passenger dropped a ball from the ceiling of the carriage, then the ball would move in a parabola relative to an observer at rest on the earth. If the train were moving with uniform velocity, the ball would appear to fall vertically downwards relative to the passenger on the train. If the train were accelerating, the ball would not move in a straight line relative to the passenger on the train, but would appear to be deflected from the vertical in a direction opposite to the direction of the acceleration of the train. If the passenger on the train wanted to interpret this motion in terms of Newton's laws of motion, he would have to introduce the fictitious inertial force $-m\mathbf{a}_r$ in order to account for the deflection of the ball from the vertical. This fictitious force produces no acceleration relative to the fixed stars. Another example of linear acceleration is a mass resting on the floor of a lift which is itself falling freely under gravity.

A different type of accelerated motion is shown in *Figure 1.3(a)*. A particle P, of mass m, is attached by a string to the point O, and is rotating on a smooth horizontal table with uniform angular velocity ω about an axis through O, as shown in *Figure 1.3(a)*. Relative to the earth, which will be considered as an approximate inertial frame, the particle P has an acceleration $r\omega^2$ directed towards O. In order to give rise to this acceleration there must be a force equal to $mr\omega^2$ acting on the particle P. This centripetal force is due to the pull of the string on the mass m. According to Newton's

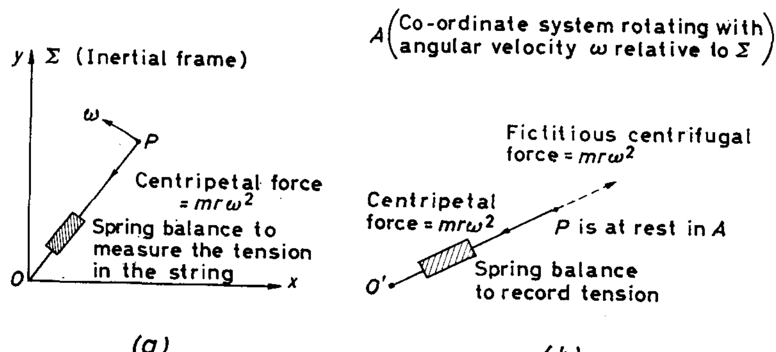

Figure 1.3. (a) Particle P rotates with uniform angular velocity ω, on a smooth horizontal table; (b) in the co-ordinate system A which rotates with P, a fictitious centrifugal force $mr\omega^2$ must be introduced if one wants to apply Newton's laws of motion in the rotating co-ordinate system

third law there is an equal and opposite force on the string. This force arises from the fact that the particle P is tending to carry on in a straight line relative to the earth, that is, it is tending to fly off at a tangent to the circular motion. The string keeps the particle P moving in a circle. The tension in the string can be measured by a spring balance. Now consider a co-ordinate system, denoted A, rotating with the particle P, such that, relative to A, P is at rest as shown in *Figure 1.3(b)*. There is still a centripetal force equal to $mr\omega^2$ acting on P and directed towards O due to the tension in the string; this force will still be measured by the spring balance. If one wanted to apply Newton's laws of motion in the rotating co-ordinate system one would have to introduce a fictitious inertial force $-mr\omega^2$ acting outwards; there would then be no resultant 'force' on P, and P would remain at rest in the rotating co-ordinate system. This fictitious outward force is sometimes called the centrifugal force. It produces no acceleration relative to the fixed stars.

vacuum, where there was nothing external or sensible with which the globes could be compared. . . .

According to Einstein's theory of general relativity, the laws of motion (in a gravitational field) can be expressed in a form which is the same in all reference frames, whether they are accelerating or not. Apart from Chapter 9, this book is concerned almost exclusively with the theory of special relativity, which deals only with inertial frames of reference.

1.6. INERTIAL REFERENCE FRAMES

The question arises, how can one select a co-ordinate system in which Newton's laws of motion are valid? If in a reference frame a particle under the influence of no forces (i.e. a particle far away from any other particles capable of exerting forces) travels in a straight line with constant speed, then Newton's first law is valid in this reference frame. This reference frame is then considered a suitable reference frame for the application of Newton's laws of motion. Such a reference frame is called an *inertial reference frame*, or sometimes a Galilean reference frame. Now it is not always possible to remove all other bodies, so the concept of inertial frames must be examined a little more closely, especially within the context of the theory of special relativity. It is generally only necessary to use the theory of special relativity when the velocities of the particles are comparable with the velocity of light. The only particles, available in the laboratory, which have such speeds are atomic particles. For atomic particles the electric forces are very much greater than the gravitational force of attraction; for example, the ratio of the electric to the gravitational force between two α-particles is $\sim 10^{35}$. Hence, for atomic particles, if all neighbouring bodies are electrically neutral and produce no magnetic fields, the effects of neighbouring bodies are negligibly small. It is possible to carry out experiments on the motions of ions in the absence of electric and magnetic fields and it is found experimentally that the ions do travel in straight lines (subject of course to the limitations due to the Uncertainty Principle and to the small gravitational attraction of the earth which can generally be neglected). Hence, for most atomic and nuclear processes studied in the laboratory, the co-ordinate system in which the earth is at rest, that is the laboratory system, is a sufficiently good approximation to an ideal inertial reference frame. For low velocity mechanical phenomena the reference system in which the earth is at rest is a satisfactory approximation to an inertial frame, provided the force of gravity

is considered as an impressed force. In this system, generally called the laboratory system, Newton's laws can be used to make accurate predictions. However, the earth is rotating about its axis and about the sun. In very accurate works these rotations have to be taken into account, e.g. with Foucault's pendulum experiment. A better approximation to an inertial frame is the reference frame at rest relative to the 'fixed' stars. These stars may have some irregular motions, but this reference frame, or one moving with uniform velocity relative to it, is a sufficiently good approximation to an ideal inertial frame for all terrestrial phenomena. For cosmological problems the choice of one single inertial frame to describe the whole universe is not appropriate, and the concept of frames of reference must be refined and extended within the context of the general theory of relativity.

PROBLEMS

Problem 1.1—Give an account of the standards of mass, length and time. How would you set up similar standards in a rocket moving with uniform velocity relative to the earth?

Problem 1.2—Distinguish between inertial mass and gravitational mass within the context of Newtonian mechanics. Discuss the experimental evidence for the proportionality of the inertial and gravitational mass of a body. (Include a discussion of Eötvös' experiment.)

Problem 1.3—If you were locked in a cave and had no contact with the outside world, what experiments, if any, could you perform to show that (*a*) the earth has a linear velocity in its orbit around the sun, and (*b*) the earth is rotating. Could you determine your geographic latitude? [Hint: For part (*b*), use either Foucault's pendulum or an accelerometer of the type described by Struve, *Elementary Astronomy*, p. 51.]

Problem 1.4—Show that Newton's laws of motion obey the principle of relativity when the co-ordinates and time are transformed according to the Galilean transformations. State clearly *all* the assumptions you make.

Problem 1.5—A stone is dropped from rest from the mast of a ship moving with a velocity of 15 m/sec. Choose the origins of Σ (the laboratory frame and Σ' (the co-ordinate system in which the ship is at rest) such that the origins coincide with the stone at the instant $t = 0$ when the stone is dropped. Show that the path of the stone relative to Σ' (the ship) is a straight line given by $x' = 0$; $y' = -gt^2$. Show that if the ship moves along the x axis, then relative to the laboratory the stone moves in a parabola given by $x = 15t, y = -gt^2$. Show how these equations are related by the Galilean transformations.

Problem 1.6—A particle is suspended from the roof of a carriage of a train moving with uniform velocity. If the particle moves in a circular motion relative to the train (e.g. a conical pendulum), use the Galilean transformations to show that its path relative to the earth is a cycloid.

Problem 1.7—Discuss what 'fictitious' inertial forces must be introduced into Newtonian mechanics, if we choose as our standard of rest a co-ordinate system rotating relative to the fixed stars.

Problem 1.8—A particle is dropped vertically towards the earth. Show that after a time t it is deviated to the east by an amount $(\omega g t^3/3) \cos \lambda$ where ω is the angular velocity of the earth, g the acceleration due to gravity and λ is the geographic latitude. (Reference: Joos, *Theoretical Physics*, 1st edition, p. 223.)

Problem 1.9—Show that the plane of vibration of a Foucault pendulum rotates with a velocity $\omega \sin \lambda$, where λ is the latitude and ω is the angular velocity of the earth. (Reference: Struve, *Elementary Astronomy*, p. 48.)

Problem 1.10—A particle is at rest relative to the earth. Relative to an observer on a rotating roundabout the particle appears to be rotating with angular velocity ω. Relative to the roundabout the fictitious centrifugal force $mr\omega^2$ acts outwards. How is the particle kept moving in a circle relative to the rotating frame of reference? [Hint: Remember the Coriolis force.]

Problem 1.11—Show that the value of the acceleration due to gravity measured by a simple pendulum at sea level varies by about 0·3 per cent between the poles and the equator due to the rotation of the earth. [Hint: Show that at the equator, when the centrifugal force is taken into account, $g \simeq g_0(1 - \omega^2 R/g_0)$, where g_0 is the acceleration due to gravity at the poles, $\omega = 7{\cdot}29 \times 10^{-5}$ rad/sec is the angular velocity and $R = 6{,}367$ km is the radius of the earth.]

2

HISTORICAL INTRODUCTION II—THE RISE OF THE THEORY OF SPECIAL RELATIVITY

2.1. INTRODUCTION

A brief review of the historical development of the theory of special relativity is given in this chapter. A fuller, more balanced account is given in the author's other text, *An Introduction to the Theory of Relativity*, (Butterworths, London, 1964), where full references to other works are given. In order to put the theory of special relativity into historical perspective, it is necessary to mention the theories of the ether developed in the nineteenth century. Details of experiments are relegated to Appendix 1. The reader should realize at the outset, that the mechanical theories of the ether turned out to be one of the biggest red herrings in the history of science. The reader should treat this chapter as a short incursion into the history of science, interpreted with hindsight and a modern bias. The reader should gloss quickly over this Chapter, and proceed in Chapter 3 to cultivate the relativistic point of view. The important fact for the reader to realize is that the theory of special relativity evolved from classical optics and electromagnetism. In the theory of special relativity, what Einstein[1] did was to extend the principle of relativity to the laws of classical optics and electromagnetism. This necessitated abandoning the Galilean transformations and re-interpreting the classical concepts of space and time. With just this amount of background, some readers may prefer to proceed directly to Chapter 3, where the theory of special relativity is developed from two main axioms, namely, the principle of relativity and the principle of the constancy of the speed of light. The whole theory can be checked *a posteriori* by showing that the predictions of the theory agree with experiment. Alternatively, one can now approach special relativity via high-speed mechanics (cf. Appendix 4).

2.2. THE LUMINIFEROUS ETHER

After the initial success of Newtonian mechanics and Newton's theory of universal gravitation, it seemed plausible to try and explain all natural phenomena in terms of mechanics, particularly

as the concepts of mechanics were familiar to people in their daily lives. This aim at a mechanical interpretation of all natural phenomena started with Descartes (born 1596) and continued up to the end of the nineteenth century. For example, towards the end of this period Lord Kelvin wrote: 'I am never content until I have constructed a mechanical model of the object I am studying. If I succeed in making one, I understand; otherwise I do not'. This approach proved very successful in the case of sound. Sound was interpreted as the propagation of longitudinal waves in a material medium. Sound will not travel through a vacuum, a material medium being essential for its transmission.

In the same way attempts were made to interpret light in terms of mechanical models, and two theories arose. One was the corpuscular theory in which light was pictured as a stream of little corpuscles. It was assumed that these small particles obeyed the laws of mechanics and produced the sensation of light when they struck the eye. The other theory was the wave theory. It was developed, initially, by Hooke, and was formulated in detail by Huygens. On this theory, light was considered as a form of wave motion propagated through space. At the time of Newton, and for some time afterwards it was assumed that the waves constituting light were longitudinal waves. It is not possible to account for the polarization of light in terms of longitudinal waves. At the time, Newton was able to account for polarization more satisfactorily in terms of the corpuscular theory. He felt that the corpuscular theory was also able to account for the rectilinear propagation of light better than the wave theory, since on the wave theory light should be diffracted around corners in the same way as sound is; the wavelength of light had not been determined at that time. The corpuscular theory remained pre-eminent until the beginning of the nineteenth century, when Young investigated the interference of two beams of light. Young and Fresnel were able to account for the newly observed phenomena of interference and diffraction on the wave theory, and from that time onwards the wave theory of light came to be accepted. Since sound will not travel through a vacuum and must have a material medium to transmit it, it was reasonable to assume that, if light was a form of wave motion, then there had to be some medium present in a vacuum as well as inside material bodies, which was able to transmit the vibrations constituting light. This medium was called the ether.

When considering the ether theories of light, it is helpful to use the theory of the transmission of sound as an analogy. In this analogy, the air or material body which transmits the sound waves

corresponds to the light wave. Now the velocity of sound in air is independent of the velocity of the source of sound, but it does depend on the strength and direction of the wind. If V is the speed of sound in still air and w is the speed of the wind, according to the Galilean velocity transformations, the speed of sound varies from $V + w$ to $V - w$ when the sound travels in the direction of the wind and in the direction opposite to it respectively. By analogy with sound, it was assumed in the ether theories that the speed of light was independent of the speed of the source of light. If the earth were moving relative to the hypothetical light transmitting medium, that is the ether, there would be an ether 'wind', relative to the earth. By analogy with sound, it was believed that this ether 'wind' should affect the velocity of light, even in empty space. If c is the speed of light when the ether is at rest and w is the speed of the ether 'wind' relative to the earth, according to the Galilean velocity transformations, eqns (1.10), (1.11) and (1.12), the speed of light should vary between $c + w$ and $c - w$ when c is parallel to w and c is antiparallel to w respectively. Thus, in principle, by measuring the speed of light in different directions in space, one should have been able to determine the motion of the earth relative to the absolute reference system in which the hypothetical ether was assumed to be at rest.

Class Question—In your course on physical optics, what type of wave motion in what type of medium did the wave equation you used represent? (Comment: At school level the answer to the above questions would probably be that the reader had been taught that light was a form of wave motion, but the question of what type of motion in what type of medium was probably never considered. By analogy with sound and water waves, such a reader would probably accept that the speed of light should be independent of the speed of the source. More advanced readers will probably think of light as an electromagnetic wave. In this case also the reader would accept that the speed of electromagnetic waves is independent of the speed of the source, cf. Appendix 2.)

Many readers may prefer not to think in terms of a mechanical light transmitting ether. Such readers can identify the hypothetical absolute system simply as a co-ordinate system in which the speed of light in empty space is the same in all directions, as shown in *Figure 2.1*(a). According to the Galilean velocity transformations, in any other reference frame moving with uniform velocity relative to this absolute system, the speed of light in empty space should be different in different directions, as shown in *Figure 2.1*(b). Due to its rotation about its axis and about the sun, it is unlikely that the earth is always at rest relative to such an absolute system, if such a

system exists. Thus by measuring the speed of light in different directions, one should have been able to determine the motion of the earth relative to the hypothetical absolute system. The most famous of these optical experiments was the Michelson–Morley experiment (1887). A full account of this experiment is given in

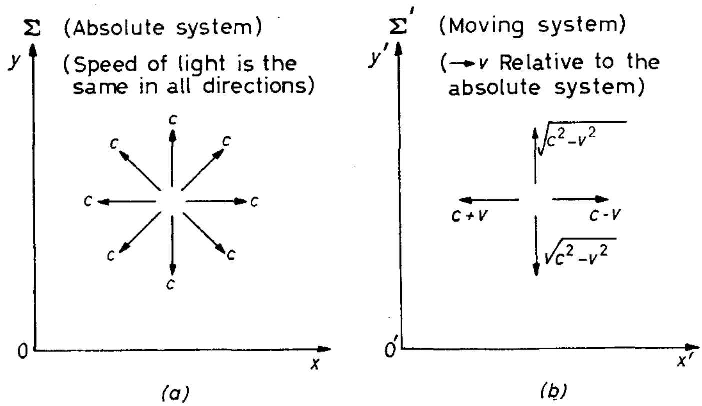

Figure 2.1. It is assumed that there is an absolute system Σ, in which the speed of light in empty space is the same in all directions, as shown in Figure 2.1(a). If it is assumed that the Galilean velocity transformations are correct, the speed of light in empty space should be different in different directions in the moving system Σ', shown in Figure 2.1(b). For example, for light parallel to the $+x$ axis in Σ, $u_x = +c$. Hence, from eqn (1.10), $u_x' = u_x - v = c - v$. For light in the negative ex direction in Σ, $u_x = -c$. From eqn (1.10), in Σ we have $u_x' = u_x - v = -c - v = -(c + v)$ as shown in Figure 2.1(b). Experiments failed to detect any variation in the speed of light in different directions relative to the earth. In the theory of special relativity, it is assumed that the speed of light in empty space is the same in all directions relative to Σ' as well as Σ. This means abandoning the Galilean transformations. This postulate is reasonable, if the laws of optics obey the principle of relativity, since there is then nothing to differentiate Σ' from Σ, if the speed of light is independent of the speed of the source, and if space is isotropic. Thus the speed of light in empty space should just as likely be the same in all directions in Σ' as in Σ. Predictions based on the assumption that the speed of light in empty space is the same in all directions in both Σ and Σ' have proved to be in accord with experiment

Appendix 1. All these optical experiments failed to measure any variation in the speed of light in different directions of empty space, and failed to identify any absolute system for the laws of optics. Various ingenious interpretations of the null result of the Michelson–Morley experiment were suggested at the time. They are reviewed in Appendix 1. The interpretation which has come to be accepted, following the development of the theory of special relativity, is that the speed of light in empty space is the same in all directions in all inertial reference frames. (This assumes that the speed of light is

independent of the speed of the source.) For example, according to this interpretation the speed of light should be the same in all directions in empty space in both *Figures 2.1*(*a*) and *2.1*(*b*). This assumption means that the Galilean transformations must be abandoned, and it leads to the re-interpretation of the classical ideas of space and time developed in Chapter 3. The failure to determine any absolute reference frame for the laws of optics was interpreted by Einstein[1], by postulating that the laws of optics obeyed the principle of relativity. For example, according to Einstein, if one were in a ship moving with uniform velocity, relative to the earth, and if one carried out optical experiments, one should develop the same laws of optics on the basis of these experiments, as if the experiments were carried out in a laboratory at rest relative to the earth. It should be impossible to determine the speed of the ship by means of optical experiments, without looking at something external to the ship. If the speed of light in empty space is independent of the speed of the source of light, and provided space is isotropic (that is, all directions are equivalent) and if there is no absolute system for the laws of optics, the speed of light in empty space should be just as likely to be in the same in all directions relative to the ship as relative to the earth. Furthermore, provided the same units of length and time are used, the numerical value of the speed of light in empty space should be the same relative to both the ship and the earth. If the speed of light in empty space relative to the ship depended on the speed of the ship relative to the earth, one could carry out experiments on the ship to see how the speed of light depended on the speed of the ship. Thereafter, by measuring the value of the speed of light relative to a moving ship, one could determine the speed of the ship relative to the earth, without looking at anything external to the ship. This is contrary to the assumption that the laws of optics obey the principle of relativity. Hence the speed of light in empty space should have the same numerical value relative to the earth and the ship. This is the principle of the constancy of the speed of light. This is one of the axioms of the theory of special relativity. Predictions based on this axiom have proved to be in agreement with experiments.

At the beginning of the twentieth century, when the theory of special relativity was developed, the wave theory of light was accepted. The old corpuscular theory, in which it was assumed that light consisted of corpuscles, which obeyed the laws of Newtonian Mechanics, had been abandoned for some time. The quantum theory of light, in which it is assumed that light consists of photons

(or light quanta) of energy $h\nu$, where h is Planck's constant and ν is the frequency of the light, was developed at the same time as the theory of special relativity. Actually, Einstein interpreted the photoelectric effect in terms of photons (light quanta) in 1905, the same year as he introduced the theory of special relativity. The resolution of the dual nature of light (wave theory and photon model) had to await the development of quantum mechanics in the 1920's. The development of the photon model of light did not influence the historical development of the theory of special relativity to any large extent. The properties of photons were shown to be consistent with the theory of special relativity. There is now direct experimental evidence to show that the speeds of photons, such as γ-rays, are independent of the speeds of their sources, (cf. Sections 3.3 and 5.8.6).

2.3. MECHANICAL THEORIES OF THE ETHER

In the nineteenth century, the transmission of electric and magnetic forces through empty space was also interpreted in terms of a hypothetical ether. Following the work of Faraday and Maxwell, it was assumed that electric and magnetic field lines gave rise to tensions in, and pressures on, elements of the hypothetical ether which transmitted, by contiguous action the forces between electric charges. The second half of the nineteenth century saw the synthesis of classical electromagnetism and optics in Maxwell's equations. Light was interpreted as an electromagnetic wave transmitted by the ether. The existence of electromagnetic waves was confirmed experimentally by Hertz. At that time, attempts were made to interpret all electric, magnetic and optical phenomena in terms of mechanical properties attributed to a single ether. In 1867, Lord Kelvin suggested that the atoms of matter were vortices in the ether. In order to be so comprehensive and all embracing in their coverage, the mechanical models of the ether became more and more complicated. To quote Born[2] (1924)

> If we were to accept them literally, the ether would be a monstrous mechanism of invisible toothed wheels, gyroscopes and gears intergripping in the most complicated fashion, and of all this confused mess nothing would be observable but a few relatively simple forces which would present themselves as an electromagnetic field.

Towards the end of the nineteenth century the view was beginning to arise that one should merely accept that the laws of electromagnetism describe the electromagnetic forces between moving

charges, and one should not try to interpret the electromagnetic forces themselves in terms of a mechanical ether, whose properties could not be measured. The view that prevailed after the rise of the theory of special relativity may be summarized by the following quotation from Born[2].

> Light or electromagnetic forces are never observable except in connection with bodies. Empty space free of all matter is no object of observation at all. All that we can ascertain is that an action starts out from one material body and arrives at another material body some time later. What occurs in the interval is purely hypothetical, or, more precisely expressed, arbitrary. This signifies that theorists may use their own judgement in equipping a vacuum with phase quantities (denoting state), fields, or similar things, with the one restriction that these quantities serve to bring changes observed with respect to material things into clear and concise relationship.
>
> This view is a new step in the direction of higher abstraction and in releasing us from common ideas that are apparently necessary components of our world of thought. At the same time, however, it is an approach to the ideal of allowing only that to be valid as constructive elements of the physical world which is directly given by experience, all superfluous pictures and analogies which originate from a state of more primitive and more unrefined experience being eliminated.
>
> From now onwards ether as a substance vanishes from theory. In its place we have the abstract 'electromagnetic field' as a mere mathematical device for conveniently describing processes in matter and their regular relationship.

The equations of electromagnetism give the electric and magnetic fields due to a system of moving charges at a point in space. From these fields the force that would act on a test charge placed at that point in empty space can be calculated using the Lorentz force. It is the effects of the forces on charges that are observed experimentally. It must be pointed out that classical electromagnetism is correct only in so far as all effects, which arise from the finite value of Planck's constant h, may be neglected. Classical electromagnetism has to be extended to incorporate the effects of quantization.

Nowadays, instead of interpreting electromagnetic forces in terms of mechanical models, we try to interpret the mechanical properties of solids and fluids in terms of atomic theory and quantum mechanics using electromagnetic forces.

For the benefit of readers familiar with quantum mechanics a very brief review will now be given of how the wave properties of light are nowadays associated with individual photons, rather than with a transmitting medium. Nowadays, photons (light quanta)

are treated as one species of the fundamental particles. According to quantum mechanics, wave properties are associated with individual photons, in a similar way to the way 'waves' are associated with individual electrons. In both cases a wave equation can be used to make estimates of the probability of finding a particle (photon or electron) at a given point. The reader will undoubtedly accept that an electron and its associated 'waves' can traverse a vacuum, without a transmitting medium. Similarly, by analogy with an electron, it is reasonable to accept that a photon, such as a γ-ray, and its associated 'waves' can cross a vacuum without a transmitting medium. Nowadays, a light transmitting medium, or ether, is generally not regarded as necessary. In the nineteenth century, it was believed that, like sound, light was a continuous wave motion in a continuous medium. It was reasonable to postulate a mechanical ether in such circumstances.

2.4. MAXWELL'S EQUATIONS AND THE PRINCIPLE OF RELATIVITY

The laws of classical electromagnetism, such as Coulomb's law, the Biot–Savart law, Faraday's law etc., are summarized by Maxwell's equations. If he has not already met them, the reader will almost certainly meet these important equations in his electricity course. In addition to describing the behaviour of electric and magnetic phenomena, Maxwell's equations can be used to interpret classical physical optics in terms of electromagnetic waves. Now, if the co-ordinates and time are transformed using the Galilean transformations, eqns (1.6)–(1.9), then Maxwell's equations do not obey the principle of relativity. It was natural in the last quarter of the nineteenth century to assume that Newtonian mechanics and the Galilean transformations were correct, since, at that time, no experiments on very high speed atomic particles had been carried out, so that no significant deviations from Newtonian mechanics had been recognized. However, if the Galilean transformations were assumed to be correct, Maxwell's equations could only hold in one inertial reference frame, and if this were true, it should have been possible to identify this absolute reference frame by means of electrical experiments such as the Trouton–Noble experiment. (See Appendix 1.) However, in practice, it proved impossible to identify any absolute reference frame for the laws of electromagnetism

What Einstein[1] did in the theory of special relativity, was to postulate that the laws of optics, electricity and magnetism obeyed

the principle of relativity. This meant abandoning the Galilean transformations and the concept of absolute time in favour of the Lorentz transformations. In his 1905 paper entitled *On the Electrodynamics of Moving Bodies* in which the theory of special relativity was first developed, Einstein[1] wrote

> Examples of this sort, together with the unsuccessful attempts to discover any motion of the earth relatively to the 'light medium', suggest that the phenomena of electrodynamics as well as of mechanics possess no properties corresponding to the idea of absolute rest. They suggest rather that, as has already been shown to the first order of small quantities, the same laws of electrodynamics and optics will be valid for all frames of reference for which the equations of mechanics hold good. We will raise this conjecture (the purport of which will hereafter be called the 'Principle of Relativity') to the status of a postulate, and also introduce another postulate, which is only apparently irreconcilable with the former, namely, that light is always propagated in empty space with a definite velocity c which is independent of the state of motion of the emitting body. These two postulates suffice for the attainment of a simple and consistent theory of the electrodynamics of moving bodies based on Maxwell's theory for stationary bodies. The introduction of a 'luminiferous ether' will prove to be superfluous inasmuch as the view here to be developed will not require an 'absolutely stationary space' provided with special properties, nor assign a velocity-vector to a point of the empty space in which electromagnetic processes take place.

Later on in the same paper, Einstein[1] wrote out his two main postulates explicitly, namely, the principle of relativity and the principle of the constancy of the velocity of light. To quote from Einstein's paper:

> The following reflexions are based on the principle of relativity and the principle of the constancy of the velocity of light. These two principles we define as follows:
>
> 1. The laws by which the states of physical systems undergo change are not affected, whether these changes of state be referred to the one or the other of two systems of co-ordinates in uniform translatory motion.
> 2. Any ray of light moves in the 'stationary' system of co-ordinates with the determined velocity c, whether the ray be emitted by a stationary or by a moving body.

At the time the theory of special relativity was introduced (1905) the generally accepted laws of optics and electromagnetism were Maxwell's equations. It would have been reasonable for Einstein[1], after extending the principle of relativity to optics and electromagnetism, to assume as his second postulate, that the appropriate laws of optics and electromagnetism were Maxwell's equations.

However, Einstein[1] chose the principle of the constancy of the velocity of light as his second main postulate. According to this postulate the numerical value of the speed of light is the same in all inertial reference frames. By choosing the principle of the constancy of the velocity of light as his second postulate Einstein[1] was able to analyse the experimental measurement of the times of spatially separated events in the way described in Section 3.8, before developing the Lorentz transformations. After developing the Lorentz transformations from the principle of the constancy of the velocity of light, in the same paper, Einstein[1] went on to show that, if the co-ordinates and time are changed according to the Lorentz transformations, Maxwell's equations, that is the laws of classical optics and electromagnetism, obey the principle of relativity.

For the benefit of readers familiar with Maxwell's equations, it is shown in Appendix 2 that, if it is assumed that Maxwell's equations are correct and obey the principle of relativity, then the principle of the constancy of the velocity of light follows.

The theory of special relativity evolved from classical electromagnetism, as can be seen from the title of Einstein's original paper[1], which was, *On the Electrodynamics of Moving Bodies*. Effectively, Einstein extended the principle of relativity to classical optics and electromagnetism. Following the development of the Lorentz transformations, and the abandonment of the concept of absolute time, it was necessary to modify the laws of mechanics. There is now a large amount of independent experimental evidence in favour of the new laws of relativistic mechanics. One can now abandon the traditional historical approach to the theory of special relativity, which was based on the principle of the constancy of the speed of light. The theory can now be approached via experiments on high-speed mechanics, in the way outlined in Appendix 4. Some readers may find it easier to appreciate the necessity for the changes in the classical concepts of space and time due to the theory of special relativity, by considering the experimental result that accelerated electrons tend to a limiting speed, namely the speed of light. This approach is discussed in Section 5.4.2 and Appendix 3(b). When approached via high-speed mechanics, the deviations from Newtonian mechanics are large, and it is obvious that experiments in high energy physics demand the abandonment of Newtonian mechanics in favour of the theory of special relativity. When special relativity is approached in the historical way via optics and electromagnetism, the experimental evidence is based on the absence of some small second order effects, which should have been present, if there was an absolute reference frame for the laws

of optics and electromagnetism (cf. the Michelson–Morley and the Trouton–Noble experiments described in Appendix 1). For this reason some readers may find the experimental necessity for the theory of special relativity more convincing when it is approached via high-speed mechanics (cf. Appendices 3(b) and 4, and Sections 5.4.2 and 5.8.6). The historical approach using the principle of the constancy of the speed of light will be used in Chapter 3, as it is convenient to discuss the measurement of the times of spatially separated events in terms of light signals.

The validity of the theory of special relativity in inertial reference frames depends, primarily, on the fact that the predictions of the theory are in agreement with the experimental results.

REFERENCES

[1] Einstein, A. *Annln. Phys.* **17** (1905) 891
[2] Born, M. *Einstein's Theory of Relativity* Ch. 5, 1924. London; Methuen

3

THE LORENTZ TRANSFORMATIONS

3.1. INTRODUCTION

Einstein's two main postulates in the theory of special relativity were the principle of relativity and the principle of the constancy of the velocity of light[1].

According to the *principle of relativity*, the laws of physics, including those of mechanics, optics, electromagnetism and nuclear physics are the same in all inertial reference frames.

According to the *principle of the constancy of the velocity of light* the speed of light in empty space has the same numerical value in all inertial reference frames. There is now direct experimental evidence in favour of the principle of the constancy of the velocity of light.

It must be pointed out that there are additional assumptions implicit in the theory of special relativity. For example, it is assumed that inertial frames exist and that, in such a reference frame, the motion of a body is uniform and rectilinear provided no forces act on the body. This definition is taken over from Newtonian mechanics. It is also assumed that in such a reference frame light is propagated rectilinearly and isotropically in free space. This assumes that all regions of space and all directions in space are equivalent. It is also assumed in the theory of special relativity that Euclidean geometry can be used to calculate the relationships between geometrical quantities. It is assumed that all time intervals are equivalent. The validity of these extra postulates is not known *a priori* but has to be tested experimentally. In fact, some of these postulates must be modified within the context of the theory of general relativity. The refinements due to the general theory are generally very small, and need only be introduced when the accuracy of the measurements make it necessary; they will be omitted from the present discussion until the theory of general relativity is considered in Chapter 9. For most practical purposes, either the laboratory system or a system at rest or moving with uniform velocity relative to the fixed stars is a sufficiently good approximation to an inertial frame satisfying the above postulates.

Before proceeding to develop the theory, the principle of relativity and the principle of the constancy of the velocity of light are discussed to see if the postulates are reasonable.

3.2. THE PRINCIPLE OF RELATIVITY

The principle of relativity is now extended to all the laws of physics including the laws of optics and of electromagnetism. According to the principle of relativity, if an isolated system is observed from two different inertial frames, one moving with uniform velocity relative to the other, though the observations on the system carried out in the two inertial systems yield different numerical values for some quantities, the laws derived on the basis of these observations should have the same mathematical form in both inertial frames. The laws of physics should not contain any terms referring to an absolute system. An example of the principle of relativity was given in Section 1.4, where Newton's laws of motion were shown to transform into Newton's laws under the Galilean transformations. This was illustrated by the example of a game of tennis played on the deck of a ship going out to sea. Whilst some of the measurements of an observer on the seashore would differ from those of an observer on the ship (e.g. measurements of the velocity of the ball) both observers would agree that Newton's laws held to a very good approximation in their respective rest frames, provided the velocities of the ship and the tennis ball were very much smaller than the velocity of light. Nowadays, the principle of relativity is generally accepted, and none of the serious alternatives to the theory of special relativity suggest that it does not hold. It must be emphasized that we can have theories, such as Newtonian mechanics, which satisfy the principle of relativity, but which are at variance with the principle of the constancy of the velocity of light. It is this latter principle which distinguishes the theory of special relativity from other relativistic theories. The principle of relativity says nothing about the numerical values of quantities in different inertial frames. It merely states that the laws are the same. It is the principle of the constancy of the velocity of light, in the form in which we are assuming it, which makes a precise statement about the numerical values of measurements carried out in different inertial reference frames.

As an example of the scope of the principle of relativity in the theory of special relativity, consider the ship going out to sea with *uniform* velocity on a calm day. If experiments in mechanics, optics, electromagnetism or nuclear physics were performed in a room inside the ship, according to the principle of relativity, the laws determined on the basis of these experiments would be precisely the same as if they were carried out in a laboratory at rest relative to the earth. Without looking at anything external to the ship, one

could not determine the speed of the ship by means of any of these experiments.

3.3. THE PRINCIPLE OF THE CONSTANCY OF THE VELOCITY OF LIGHT

In this section a brief review of some of the evidence available in favour of the principle of the constancy of the velocity of light will be given. If the velocity of light is to have the same numerical value in all inertial frames, then the velocity of light must be independent of the velocity of the source of light relative to the observer and must have the same value in all directions and in all regions of empty space.

The generally accepted laws of classical optics and electromagnetism are Maxwell's equations. It is shown in Appendix 2 that, if Maxwell's equations are correct and obey the principle of relativity, then the principle of the constancy of the velocity of light follows. Thus, if the accepted laws of classical optics and electromagnetism are to obey the principle of relativity, then the principle of the constancy of the velocity of light and the re-interpretation of space and time the acceptance of this principle necessitates, must be accepted.

For a long time, there was no direct experimental check of the principle of the constancy of the velocity of light. Recently, several experimental checks have been performed. (James and Sternberg[2], Sadeh[3], Alväger, Nilsson and Kjellman[4], Babcock and Bergman[5], and Alväger, Farley, Kjellman and Wallin[6].) For example, it will be described in Section 5.8.6 how Alväger and colleagues[6] determined the velocities of photons arising from the decay of π^0-mesons into two photons each. The velocity of the π^0-mesons, estimated using the equations of the theory of special relativity, was $0{\cdot}99975\ c$. The measured velocity of the photons arising from the decay of moving π^0-mesons was $(2{\cdot}9977 \pm 0{\cdot}0004) \times 10^8$ m/sec. This value is consistent with the accepted value of $2{\cdot}9979 \times 10^8$ m/sec for the velocity of light emitted by a stationary source. This result shows that the velocity of light does not add to the velocity of the source according to the Galilean transformations, eqns (1.10), (1.11) and (1.12). Thus, there is now direct experimental evidence in favour of the principle of the constancy of the velocity of light.

The statement that the speed of light has the same numerical value in all inertial reference frames implies that the fundamental units of length and time are defined in the same way in all inertial reference frames, for example, as described in Section 1.2.

There is an alternative approach to the theory of special relativity based on relativistic mechanics. This approach is outlined in Appendix 4. Instead of the principle of the constancy of the speed of light, one can start from the experimental result, described in Section 5.4.2, that the limiting speed for electrons, accelerated in an electric field, is the speed of light in empty space. If all inertial frames are equivalent, this limiting speed should have the same numerical value in all inertial frames. If this were not true, then it would be possible to separate the inertial frames in terms of the different values of the limiting speed (cf. Section 5.4.2). This postulate will be called the *principle of the constancy of the limiting speed for particles*. It is illustrated in Section 5.8.6, that, by treating light photons as particles of zero rest mass travelling at the limiting speed c, the principle of the constancy of the speed of light is a special case of the principle of the constancy of the limiting speed for particles.

3.4. THE LORENTZ TRANSFORMATIONS

It has been shown that there is direct evidence in favour of the principle of the constancy of the velocity of light. This principle, together with the principle of relativity and those relating to the isotropy of space, the rectilinear propagation of light in free space and the applicability of Euclidean geometry, will now be taken as axiomatic. Some of the consequences of the axioms are derived in Sections 3.4 and 3.5, and it is then seen if these predictions agree with the experimental results. No infallibility will be claimed for the theory of special relativity, since all theories can be replaced by new theories, provided the new theories are in *better* agreement with the experimental results. It must be stressed that one must be prepared to replace Newtonian mechanics completely. That such a radical change is necessary can be seen immediately by considering a simple example. Consider two inertial frames Σ and Σ', with Σ' moving with uniform velocity v along the common positive Ox axis relative to Σ. Let a beam of light be emitted along the common x axis at the instant $t = t' = 0$, when the origins of Σ and Σ' coincide. Let the light reach a detector at a position x in Σ at a time t. Let an observer at rest in Σ' record this event at a position x' at a time t'. Now according to the principle of the constancy of the velocity of light $x/t = c$ in Σ and $x'/t' = c$ in Σ', where c is the velocity of light. Since the origin of Σ' has moved relative to Σ, x' must be less than x. Hence, t' must be less than t, so that according to the theory of special relativity time cannot be

absolute and one cannot write $t' = t$. Thus one must be prepared to change one's intuitive ideas about absolute time, ideas that were formed in the macroscopic world we live in and in which all velocities are normally very much smaller than the velocity of light. One cannot say at the outset what all the changes in outlook must be. Initially the precise meaning of the symbols used in the theory must remain hidden in the text until the appropriate stage of the theory has been developed, such that the terms can be defined and interpreted. It was illustrated in Section 1.3, from a consideration of Newtonian mechanics, how the meaning of words such as force and mass only take meaning in terms of a particular theory. One must now be prepared to re-interpret even measurements of length and time in terms of the theory of special relativity.

Some readers may prefer to repeat the above arguments, in terms of the motion of a particle, such as an electron, moving along the x and x' axes of Σ and Σ' respectively, at speeds very close to the limiting speed c relative to *both* Σ' and Σ. If the electron passes the origins of Σ and Σ' at $t = t' = 0$, and is detected at (x, t) and (x', t') relative to Σ and Σ' respectively, then $x/t \simeq c$ and $x'/t' \simeq c$. Since the origin of Σ' moves relative to Σ, x' cannot be equal to x. Hence, t' cannot be equal to t. This illustrates, in terms of the motion of a very high energy electron, that time cannot be absolute.

According to the principle of relativity the mathematical expressions for the laws of physics should have the same form in all inertial frames of reference. In practice one needs only one such frame to describe the course of physical phenomena and for most practical purposes the laboratory frame is chosen. It is sometimes more convenient to carry out the calculations relative to another inertial frame moving with uniform velocity relative to the first. The question now discussed is, if measurements are made and the results interpreted relative to one inertial frame, then how can the numerical values of the variables relative to another inertial frame, moving with uniform velocity relative to the first, be calculated. We shall start by considering how the co-ordinates and time of an event in one inertial frame must be related to the co-ordinates and time of the same event, measured in another inertial frame moving with uniform velocity relative to the first, if the postulates of the theory of special relativity are taken as axiomatic.

It is assumed that there are two inertial frames, Σ and Σ', each having its own set of imaginary rulers so that observers at rest in either reference frame can observe the co-ordinates of events when they happen. As the time interval between two events in Σ may not be equal to the time interval between the same two events

measured in Σ', one clock reading absolute time is not adequate, so it is imagined that two sets of clocks are distributed throughout space, one set of clocks at rest relative to Σ, the other set at rest relative to Σ'. All the clocks at rest in Σ are assumed to be synchronized with each other and all the clocks at rest in Σ' are assumed to be synchronized with each other. (See Section 3.8 for a discussion of how the clocks may be synchronized.) It is assumed that all the clocks keep good time and, once synchronized, remain synchronized. [Some readers may prefer to visualize the determination of the co-ordinates and times of distant events using radar methods. Such readers can proceed first to Section 3.9 for a discussion of radar methods, then to Appendix 6 for a derivation of the Lorentz transformations and then proceed from eqn (3.7) of this Section.]

Let the zero of time in both Σ and Σ' be chosen to be the instant when the origins of Σ and Σ' coincide (at $t = t' = 0$). Let Σ' move with uniform velocity v along the common x and x' axes and let the Oy and $O'y'$ and the Oz and $O'z'$ axes coincide at $t = t' = 0$. It must be stressed that it is being assumed that the inertial frames have been moving and continue to move with uniform velocity relative to each other for all time, so that there is no acceleration of one reference frame relative to the other. A discussion of the possible effects of the acceleration of one reference frame relative to the other is deferred until Chapter 8, in which the clock paradox is discussed. It is also assumed that the measuring instruments used in Σ' are not constructed in Σ and then accelerated into Σ', but that observers in Σ and Σ' build their own apparatus from materials they find around them, using the same units to calibrate them as discussed in Section 1.2. This method overcomes any difficulties arising from possible effects due to the acceleration of measuring instruments.

Consider a beam of light emitted at the instant when the origins of Σ and Σ' coincide at $t = t' = 0$, as shown in *Figure 3.1(a)*. Let the light reach a detector at the point P. Let this event be measured using the rulers and clocks at rest in Σ and let it be recorded to be at the position x, y, z at a time t. Assuming that light is propagated rectilinearly in the inertial frame Σ, then an observer at rest in Σ would say that the light travelled along the path OP, such that

$$\frac{OP}{t} = \frac{\sqrt{x^2 + y^2 + z^2}}{t} = c$$

or

$$x^2 + y^2 + z^2 - c^2t^2 = 0 \tag{3.1}$$

where c is the velocity of light.

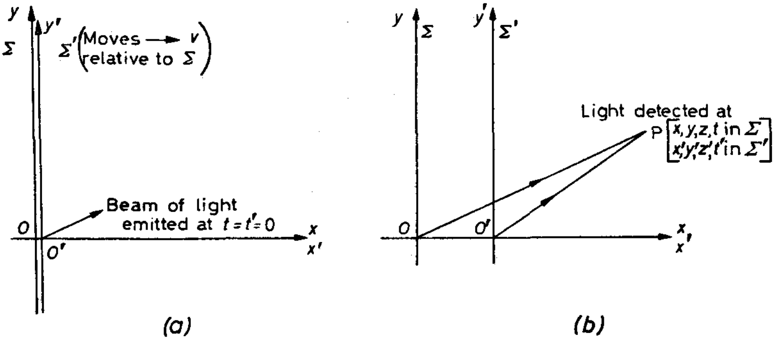

Figure 3.1. (a) A beam of light is emitted when the origins of Σ and Σ' coincide at $t = t' = 0$. This light is detected at a point P at a later time. (b) The light appears to go along the path OP relative to Σ, and along the path O′P relative to Σ'

An observer at rest in Σ' would agree that the light did reach the detector (which may be moving relative to both Σ and Σ'). Using rulers and clocks stationary in Σ', let the observer at rest in Σ' record this event at a position x', y', z' at a time t'. Relative to Σ' the light would appear to have travelled the path $O'P$ as shown in *Figure 3.1*(*b*) such that

$$\frac{O'P}{t'} = \frac{\sqrt{x'^2 + y'^2 + z'^2}}{t'} = c \tag{3.2}$$

or

$$x'^2 + y'^2 + z'^2 - c^2t'^2 = 0 \tag{3.3}$$

Notice that the same value was used for the velocity of light in both Σ and Σ', so as to be in accord with the principle of the constancy of the velocity of light. The co-ordinates x, y, z and t in Σ, and x', y', z' and t' in Σ' refer to the *same* event, namely the detection of the light at P in *Figure 3.1*(*b*). The appropriate co-ordinate transformations must transform eqn (3.3) into eqn (3.1). Direct substitution in eqns (3.1) and (3.3) will confirm that the correct transformations are

$$x' = \frac{(x - vt)}{\sqrt{1 - v^2/c^2}} = \gamma(x - vt) \tag{3.4}$$

$$y' = y \tag{3.5}$$

$$z' = z \tag{3.6}$$

$$t' = \frac{(t - vx/c^2)}{\sqrt{1 - v^2/c^2}} = \gamma(t - vx/c^2) \tag{3.7}$$

These are the Lorentz transformations. A derivation is given in Appendix 3(a). An alternative derivation of the Lorentz transformations, also based on the principle of the constancy of the speed of light, but using radar methods to determine the co-ordinates and times of events is given in Appendix 6. Some readers may find these radar methods easier to follow than using imaginary rulers and synchronized clocks distributed throughout space

Alternatively, the Lorentz transformations may be developed from the experimental result, described in Section 5.4.2, that the speed of light in empty space is the limiting speed for electrons. A short proof is given in Appendix 3(b).

The inverse transformations are

$$x = \frac{(x' + vt')}{\sqrt{1 - v^2/c^2}} = \gamma(x' + vt') \tag{3.8}$$

$$y = y' \tag{3.9}$$

$$z = z' \tag{3.10}$$

$$t = \frac{(t' + vx'/c^2)}{\sqrt{1 - v^2/c^2}} = \gamma(t' + vx'/c^2) \tag{3.11}$$

Notice that the inverse transformations can be obtained from eqns (3.4)–(3.7) by interchanging primed and unprimed quantities and replacing v by $-v$. (The inverses of all relativistic transformations can be obtained by this procedure.) Eqns (3.4)–(3.7) are generally known as the Lorentz transformations. From time to time we shall use the abbreviation

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \tag{3.12}$$

where v is the velocity of one inertial frame relative to the other. Some writers use β for $1/(1 - v^2/c^2)^{\frac{1}{2}}$. However, the symbol β is used extensively for the ratio of the velocity of a particle to the velocity of light, particularly in high energy nuclear physics where the theory of special relativity is used extensively. For this reason we shall only use the symbol γ for $1/(1 - v^2/c^2)^{\frac{1}{2}}$. It must be stressed that the Lorentz transformations give the relations between the co-ordinates and time of the same event measured in Σ and Σ' respectively. By an event is meant something that happens independently of the choice of any particular co-ordinate system, that is, something which observers in all inertial frames will agree happens, such as the collision of two particles. Observers in all frames of reference will agree that a collision between two particles takes place.

Eqns (3.1) and (3.3) relate to the co-ordinates and time of the event of detection of the light at the point P shown in *Figure 3.1*. The light could be detected by a photomultiplier moving relative to both Σ and Σ'. Observers in Σ and Σ' would agree that the light gave rise to an electrical impulse in the photomultiplier but they would attribute different co-ordinates and times to the event.

The calculation of γ can often be simplified by using a trigonometrical method. Let $v/c = \sin\theta$. Then,

$$\gamma = \frac{1}{\sqrt{1 - \sin^2\theta}} = \frac{1}{\cos\theta} = \sec\theta$$

For example, if $v/c = 0{\cdot}8000$, using four figure tables one finds that θ is 53°8′ and $\sec\theta$ is $1{\cdot}6668$. By direct calculation γ is $\frac{5}{3}$.

If $v \ll c$, then $\gamma = (1 - v^2/c^2)^{-\frac{1}{2}}$ is approximately unity and the Lorentz transformations reduce to the Galilean transformations, provided $vx/c^2 \ll t$.

So far only one inertial frame, Σ', has been considered, moving with uniform velocity v relative to Σ. Now consider another co-ordinate system Σ'' moving with uniform velocity w relative to Σ' along the common x axis; Σ'' is also an inertial frame. According to the Lorentz transformations

$$x'' = \frac{(x' - wt')}{\sqrt{1 - w^2/c^2}}; \quad y'' = y'; \quad z'' = z'; \quad t'' = \frac{(t' - wx'/c^2)}{\sqrt{1 - w^2/c^2}} \qquad (3.13)$$

If Σ' moves with uniform velocity v relative to Σ along the common x axis, one has

$$x' = \frac{(x - vt)}{\sqrt{1 - v^2/c^2}}; \quad y' = y; \quad z' = z; \quad t' = \frac{(t - vx/c^2)}{\sqrt{1 - v^2/c^2}} \qquad (3.14)$$

Substituting for x', y', z' and t' from eqns (3.14) into eqns (3.13) one obtains

$$x'' = \frac{(x - vt) - w(t - vx/c^2)}{\sqrt{(1 - v^2/c^2)(1 - w^2/c^2)}}$$

$$= \frac{x(1 + vw/c^2) - t(w + v)}{\sqrt{(1 - v^2/c^2)(1 - w^2/c^2)}}$$

Dividing top and bottom by $(1 + vw/c^2)$:

$$x'' = \frac{x - [(w + v)/(1 + wv/c^2)]t}{\sqrt{[(1 - v^2/c^2)(1 - w^2/c^2)]/(1 + wv/c^2)^2}}$$

Now the square of the denominator is equal to

$$\frac{1 - v^2/c^2 - w^2/c^2 + w^2v^2/c^4}{(1 + wv/c^2)^2} =$$

$$\frac{(1 + wv/c^2)^2 - \{(v^2/c^2) + (w^2/c^2) + (2wv/c^2)\}}{(1 + wv/c^2)^2} = 1 - \frac{(w + v)^2}{c^2(1 + wv/c^2)^2}$$

Hence

$$x'' = \frac{x - Vt}{\sqrt{1 - V^2/c^2}}$$

where

$$V = \frac{w + v}{1 + wv/c^2}$$

Similarly,

$$y'' = y;\ z'' = z \text{ and } t'' = \frac{t - Vx/c^2}{\sqrt{1 - V^2/c^2}}$$

Therefore two successive Lorentz transformations *in the same direction* are equivalent to a single Lorentz transformation, but the relative velocity of the origins of Σ and Σ'' is equal to

$$(w + v)/(1 + wv/c^2)$$

and not $w + v$, as would be the case if the Galilean velocity transformations were correct. If there were a particle at rest at the origin of Σ'', then it would be moving with velocity w relative to Σ' and velocity $(w + v)/(1 + wv/c^2)$ relative to Σ, hence a velocity u'_x along the x' axis of Σ' is expected to transform into a velocity equal to $(u'_x + v)/(1 + vu'_x/c^2)$ relative to Σ.

In this section the Lorentz transformations were derived from the principle of relativity and the principle of the constancy of the velocity of light. Some of the implications of the Lorentz transformations will now be discussed.

3.5. SOME MATHEMATICAL CONSEQUENCES OF THE LORENTZ TRANSFORMATIONS

Now, without stopping to think too much at present about the physical implications of the theory, and leaving aside the precise meaning of the symbols, some of the mathematical consequences of the Lorentz transformations are worked out and their implications are then considered in Sections 3.6, 3.7 and 3.10.

3.5.1. *Relativity of Simultaneity of Events*

Let two events occur at two separated points x_1 and x_2 in the inertial frame Σ. Let them be measured to occur at the same time

t in Σ. According to the Lorentz transformations these would be recorded at times t_1' and t_2' by clocks at rest in Σ', where t_1' and t_2' are given by

$$t_1' = \gamma\left[t - \frac{v}{c^2}x_1\right] \quad \text{and} \quad t_2' = \gamma\left[t - \frac{v}{c^2}x_2\right]$$

Since x_1 is not equal to x_2, t_1' cannot be equal to t_2', so that if the Lorentz transformations are correct, then two spatially separated events which are simultaneous in Σ, would not be measured to be simultaneous in Σ'. Similarly, if two events occur simultaneously at two spatially separated points x_1' and x_2' in Σ', according to the Lorentz transformations they would not be simultaneous in Σ. Thus, according to the theory of special relativity the simultaneity of spatially separated events is not an absolute property as it was assumed to be in Newtonian mechanics.

3.5.2. *Time Dilation*

Let a clock at rest at the point x in Σ, as shown in *Figure 3.2(a)*, give out signals, at times t_1 and t_2, such that the interval between the

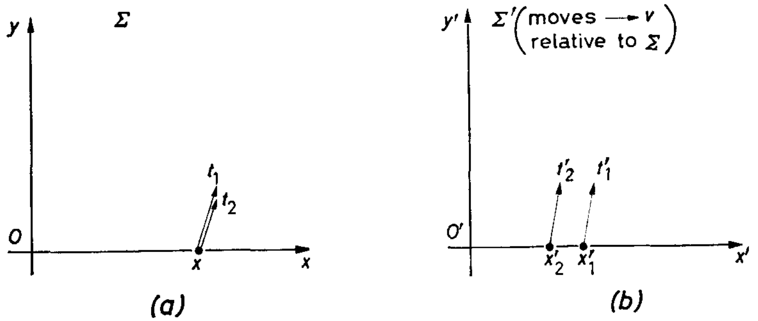

Figure 3.2. (a) In Σ the two events occur at the same point x. (b) In Σ' they do not occur at the same point

times of emission of the signals measured in Σ is given by

$$\Delta t = t_2 - t_1 \tag{3.15}$$

Let the co-ordinates and times of these two events relative to Σ' be x_1' and x_2' and t_1' and t_2' respectively, as shown in *Figure 3.2(b)*. The time interval between the events in Σ' is equal to

$$\Delta t' = t_2' - t_1'$$

Using the Lorentz transformations,

$$\Delta t' = \gamma\left[t_2 - \frac{vx}{c^2}\right] - \gamma\left[t_1 - \frac{vx}{c^2}\right] = \gamma[t_2 - t_1]$$

or

$$\Delta t' = \frac{\Delta t}{\sqrt{(1 - v^2/c^2)}} \tag{3.16}$$

According to the theory of special relativity the time interval between the two events should be longer in Σ' than in Σ. This is the phenomenon of time dilation. Notice that only one clock at the point x was necessary to measure the time interval Δt in Σ, whereas, since $x_1' = \gamma(x - vt_1)$ is not equal to $x_2' = \gamma(x - vt_2)$, two clocks at x_1' and x_2' were necessary to measure the time interval $\Delta t'$ in Σ'. This brings in the problem of synchronizing spatially separated clocks in Σ'. This point is discussed in detail in Section 3.8. The time interval between two events taking place at the same point in an inertial frame and measured by one clock stationary at that point is called the *proper* time interval between the two events. If the theory of special relativity is correct, it necessitates a revision of the Newtonian concept of absolute time, though if $v \ll c$, then $\Delta t' \simeq \Delta t$. Hence the deviations from absolute time are very small in our normal daily lives.

If signals were given out at times t_1' and t_2' by a clock *at rest* at a point x' in Σ', then in Σ these events would be recorded at times t_1 and t_2, such that

$$t_2 - t_1 = \gamma[t_2' + vx'/c^2] - \gamma[t_1' + vx'/c^2]$$

or

$$\Delta t = \frac{\Delta t'}{\sqrt{1 - v^2/c^2}} \tag{3.17}$$

In this case $\Delta t'$ is the proper time interval and is measured by one clock, whereas Δt is now an improper time interval and has to be measured by two separated clocks in Σ. Comparing eqn (3.16) with eqn (3.17) it will be seen that, according to the Lorentz transformations, proper time intervals in either Σ or Σ' should be measured to be dilated by a factor γ in the other inertial frame.

3.5.3. *The Lorentz–FitzGerald Contraction*

In order to measure the length of a stationary object, in principle, one could lay the object on a ruler and then look successively at the positions of each end of the object. If the object were moving, then both ends would have to be measured at the same time. This point is brought out vividly by an example given by Lieber and Lieber[7] and illustrated in *Figure 3.3.* If a fish is swimming with velocity v relative to the laboratory frame (which will be identified with Σ), one would not measure the position of the fish's tail at the point B

and then wait until it had swum until its mouth was opposite the point C before measuring the position of the other end of the fish. Not even the most fervent angler would stoop to that but would insist that one should measure both ends of the moving fish at the same time relative to the laboratory. The reference frame moving with the fish will be identified with Σ'.

Let the positions of the ends of the fish be measured at the same time t in Σ and recorded to be at the points x_1 and x_2 as shown in *Figure 3.3(a)*. The fish would record the events associated with the

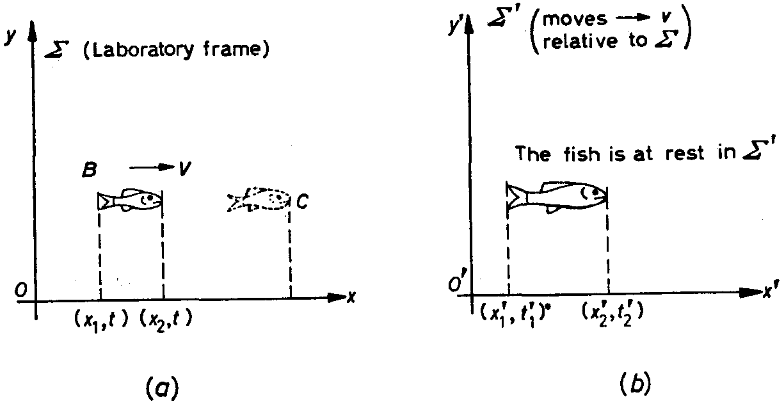

Figure 3.3. (a) The measurement of the length of a fish which is swimming with uniform velocity v relative to Σ. (b) In the inertial frame Σ', the fish is at rest

measurement of its length at positions $x_1' = \gamma[x_1 - vt]$ and $x_2' = \gamma[x_2 - vt]$ at times $t_1' = \gamma[t - vx_1/c^2]$ and $t_2' = \gamma[t - vx_2/c^2]$ respectively. As the fish does not move in Σ', its extremities can be measured at any time in Σ' so that, even though t_1' is not equal to t_2', the length of the fish measured in Σ' is equal to $x_2' - x_1'$. Hence,

$$x_2' - x_1' = \gamma(x_2 - x_1)$$

or

$$(x_2 - x_1) = \sqrt{1 - v^2/c^2}\,(x_2' - x_1')$$

Hence,

$$l = l_0\sqrt{1 - v^2/c^2} \tag{3.18}$$

where $l_0 = (x_2' - x_1')$ is the length of the fish measured in the coordinate system in which it is at rest; l_0 is called the *proper* length of the fish, whilst $l = (x_2 - x_1)$ is the length of the fish measured in an inertial reference frame relative to which the fish is moving with uniform velocity v. This is the phenomenon of length contraction.

Since t_2' is not equal to t_1' the fish would 'say' that according to 'its' clocks the measurements were not carried out at the same time. For example, let two observers, at rest in the laboratory system carry out the actual experiment of measuring the length of the fish by noting, at the same time t, the positions of the front and rear of the fish on a ruler, which is stationary in the laboratory system. The fish would 'claim' that, relative to the reference frame Σ' in which it was at rest, one observer measured the front of its nose at a time t_2', and the ruler was then moved with velocity $-v$ until a later time t_1', when the second observer measured the position of its tail. The fish would 'say', 'No wonder the laboratory observers measured my length to be too short. They moved the ruler'. It can be seen that the measurement of length involves the question of the simultaneity of spatially separated events.

Since according to the Lorentz transformations $y = y'$ and $z = z'$, the dimensions of moving bodies perpendicular to their direction of motion are not measured to be Lorentz contracted.

The phenomenon of length contraction is reciprocal. If the fish measured the length of an object, which was stationary in the laboratory, at the same time t' in Σ' and found the extremities of the object to be at the points x_2' and x_1', then, according to the Lorentz transformations, one would have

$$x_2 - x_1 = \gamma[x_2' + vt'] - \gamma[x_1' + vt']$$

or

$$x_2' - x_1' = \sqrt{1 - v^2/c^2}(x_2 - x_1)$$

Hence, in general, the *measured* length of an object is less in its direction of motion, when the object is moving relative to an observer, compared with the length of the object measured by an observer at rest relative to the object. Notice it was said that a moving object is *measured* to contract in the direction of motion, in the sense that one observes the positions of the ends of its extremities on a ruler at the same time. What one sees (or photographs) depends on the light actually reaching the eye (or camera) at that instant, and the light from different parts of the object will have left the object at different times depending on the distances of the various parts of the object from the observer. What one sees depends also on the Doppler effect. These effects are discussed in Section 4.4.

Let a body of proper volume V_0 move with uniform velocity v relative to Σ. If it is divided into thin rods parallel to v, since each one of these is reduced in length by an amount given by eqn (3.18), the measured total volume of the body relative to Σ should be

$$V = V_0\sqrt{1 - v^2/c^2} \tag{3.19}$$

It has been shown in this section that if the theory of special relativity is correct then it necessitates the abandonment of the concepts of absolute length and absolute time. The above results are developed using the K-calculus in Appendix 6.

3.6. THE SIMULTANEITY OF SPATIALLY SEPARATED EVENTS

The criterion for accepting a theory is whether or not the predictions of the theory are in agreement with the experimental results. In this section, it is discussed qualitatively whether the predictions of the theory of special relativity about length contraction and time dilation are plausible and in accord with experience. Before lengths can be measured and spatially separated clocks synchronized, one must decide what is meant by the simultaneity of spatially separated events.

It would be easy to say which of two events took place first, if the events occurred at the same place. If the events were separated in space, then signals would have to be sent out when the events happened. These signals would reach the observer at a later time. If the observer knew the speed of propagation of the signals and the positions of the events, then he could estimate when the events took place. Ideally, if signals could be transmitted with infinite velocity, such that they would reach the observer without time delay, then the observer need only note the time of arrival of the signals and equate it to the time of the event. However, there is no known way of transmitting signals instantaneously; all actual signals take a finite time to go from the event to the observer. The fastest signals available are light signals. It will be assumed, in the present discussion, that light signals are emitted when events happen. The light signals then travel to the observer who detects the signals with a light detector, such as a photomultiplier. If the observer records the signal at a time t and measures that the event was a distance r away from him, then he would estimate that the event took place at a time $t - r/c$, where c is the velocity of light.

In order to overcome the necessity of measuring time intervals Einstein[8] considered an example in which the light signals from two events were observed at a point midway between the events. If the events occurred at A and B respectively and emitted light signals when they occurred, and if the light signals from A and B reached a point C half way between A and B at the same time, then it would be reasonable to assume that the events occurred simultaneously at a time $AB/2c$ before the light signals reached C. This is a satisfactory definition of simultaneity in this particular case,

provided the velocity of light is the same in both directions and provided light travels in straight lines.

Let two rockets denoted 1 and 2 pass each other. Let both rockets move with uniform velocity relative to the fixed stars so that the two co-ordinate systems, in which one or other of the rockets is at rest, are both inertial systems. Let the relative velocity of the rockets be v. The reference system in which rocket 1 is at rest is identified with Σ and the inertial system in which rocket 2 is at rest is identified with Σ'. It is assumed that there are observers in both rockets and that these observers have devices, such as photomultipliers, to detect light signals coming from different directions and also measuring rods to measure distances.

It is assumed that two lightning thunderbolts strike both rockets just as the rockets are passing each other. Let the thunderbolts leave marks on both rockets. These are two events independent of any co-ordinate system, since the observers on both rockets would agree that thunderbolts did strike on two occasions. Let the positions of the marks caused by the lightning on rockets 1 and 2 be denoted by A and A' and B and B' respectively, and let the light detectors on rockets 1 and 2 be at C and C' respectively, as shown in *Figure 3.4*. It will be assumed that each observer finds afterwards that his light detectors were exactly half way between the marks left on his rocket, that is $AC = CB$ on rocket 1 and $A'C' = C'B'$ on rocket 2. The measuring sticks need not even be calibrated in the same units to show these equalities. The length measurements can be carried out by the observers on the rockets at their leisure, by walking along the rockets.

It is assumed that the light signals from A and B arrive at the same time at the detectors at C as shown in *Figure 3.4(c)*. If the observer on rocket 1 assumes that the velocity of light in empty space is independent of the velocity of the source and is the same in all directions in space, as predicted by Maxwell's equations, and the principle of the constancy of the velocity of light, then he would conclude that the two lightning flashes struck simultaneously at A and B at a time AC/c before the light signals reach C.

At the instant the lightning strikes at A, then A coincides with A', and when it strikes at B, then B coincides with B'. Since the lightning strikes at the same time at A and B relative to rocket 1, and since $AC = CB$ and $A'C' = C'B'$, it is reasonable to assume that, relative to Σ, C coincides with C' when the lightning strikes at A and B as shown in *Figure 3.4(a)*, provided all times are measured relative to rocket 1 (Σ). The light emitted by the lightning takes a finite time to reach the detectors at C and C', and during this

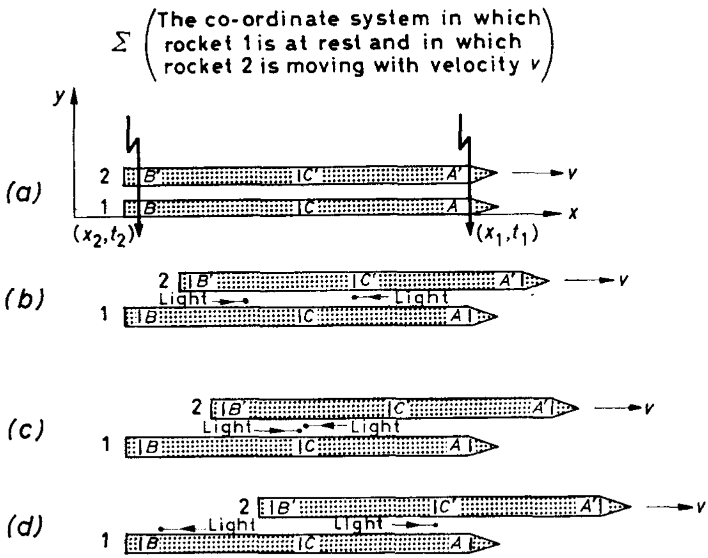

Figure 3.4. (*a*) *Two rockets labelled* 1 *and* 2 *are passing each other when lightning strikes twice leaving marks on both rockets. The lightning strikes simultaneously in the co-ordinate system* Σ *in which rocket* 1 *is at rest, such that the light signals from A, A′ and B, B′ reach the light detector at C, situated halfway between A and B, simultaneously as shown in* (*c*). *The light signal from A, A′ passes C′ before the light from B, B′ does, as illustrated in* (*b*) *and* (*d*) *respectively*

time interval rocket 2 moves relative to rocket 1 such that C' moves towards A, as illustrated in *Figure 3.4*(b). Thus the light travelling from A towards C must pass C' before reaching C, as shown in *Figure 3.4*(b). A little later, relative to Σ, the light from the flashes at A and B reach C simultaneously, as shown in *Figure 3.4*(c), so that the observer on rocket 1 concludes that the lightning struck at the same time at A and B. Notice the light from B and B' does not reach C' until after it has passed C as shown in *Figure 3.4*(d). As a result the observer on rocket 2 would record the signals from A and A' before the signals from B and B', say at times t' and $t' + \Delta t'$ respectively. The observer on rocket 2 measures $A'C'$ to be equal to $C'B'$. If he assumes that the velocity of light is the same in both directions, and he is just as entitled to do so as the observer on rocket 1, then he estimates that the events at A' and B' occurred at times $t' - A'C'/c'$ and $t' + \Delta t' - B'C'/c'$ respectively, where c' is the velocity of light relative to rocket 2. The observer on rocket 2 would conclude that the events did not occur simultaneously in his reference frame, and would conclude that the event

A' happened before the event B'. Both observers use exactly the same criterion for the simultaneity of spatially separated events. It is quite possible for the lightning to strike such that the light signals reach C' simultaneously, in which case the observer on rocket 2 would record the events as simultaneous; but in this case the light from B will pass C before reaching C', whilst the light from A reaches C at a later time so that the observer on rocket 1 would record that the lightning struck at B before A. When the problem of the simultaneity of spatially separated events is related to measurements that can actually be performed, then, if the velocity of light in empty space is independent of the velocity of the source relative to the observer, and is the same in all directions, it can then be seen that events measured to be simultaneous in one inertial frame are not measured as simultaneous in another inertial frame moving with uniform velocity relative to the first.

Notice that in *Figure 3.4* the relative velocity of the two rockets was comparable with the speed of light. Normally rockets go rather slower than this so that the differences from absolute simultaneity would normally be too small to be noticed in our daily lives.

The Lorentz transformations can be used to fill in the mathematical background to the above qualitative discussion. Let the event of the lightning's striking at A, A' be recorded at (x_1, t_1) relative to Σ, the inertial reference frame shown in *Figure 3.4* and in which rocket 1 is at rest. Let the same event at A, A' be recorded at (x_1', t_1') relative to Σ', the inertial reference frame in which rocket 2 is at rest as shown later in *Figure 3.5*. Let the event at B, B' be recorded at (x_2, t_2) and (x_2', t_2') relative to Σ and Σ' respectively. Applying the Lorentz transformations,

$$t_2' = \gamma(t_2 - vx_2/c^2)$$

and

$$t_1' = \gamma(t_1 - vx_1/c^2)$$

If the events are simultaneous relative to Σ, $t_1 = t_2$. Hence,

$$t_2' - t_1' = \gamma v(x_1 - x_2)/c^2$$

In *Figure 3.4(a)*, $x_1 > x_2$ so that t_2' is greater than t_1'. Hence relative to the inertial frame Σ' in which rocket 2 is at rest, the event at B, B' should be measured to occur after the event at A, A', as predicted previously by our qualitative discussion.

It will be assumed that the lightning marks are at the ends of the rockets. Relative to Σ, rocket 2 is moving with uniform velocity v. According to the Lorentz transformations, relative to Σ, rocket 2 should be measured to be Lorentz contracted such that its measured

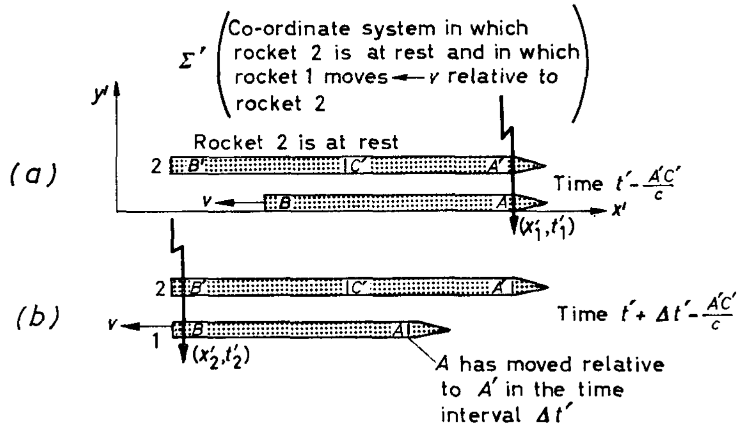

Figure 3.5. Co-ordinate system Σ' in which rocket 2 is at rest, rocket 1 moves to the left with uniform velocity v. (a) Since the light signal from A, A' reaches C' before the light signal from B, B', it is calculated that the lightning struck at A, A' before B, B' relative to Σ'. (b) Since rocket 1 moves relative to Σ' in the time interval between these two events, it is concluded that in Σ', AB is less than $A'B'$. In Σ AB is equal to $A'B'$ as shown in Figure 3.4.

length, relative to Σ, is $(l_0)_2\sqrt{1 - v^2/c^2}$, where $(l_0)_2$ is the proper length of rocket 2 measured in the inertial reference frame Σ' in which it is at rest. Since rocket 1 is at rest relative to Σ, its measured length is its proper length $(l_0)_1$. Relative to Σ, *Figure 3.4*(*a*), the *measured* lengths of the rockets are the same, since, relative to Σ, A coincides with A' when B coincides with B'.

Hence

$$(l_0)_1 = (l_0)_2\sqrt{1 - v^2/c^2} = (l_0)_2/\gamma \tag{3.20}$$

It is of interest to note that $(l_0)_2$ the proper length of rocket 2 is greater than $(l_0)_1$ the proper length of rocket 1.

3.7. THE LENGTH OF A MOVING OBJECT

As an example of the measurement of the length of a moving object in a direction parallel to its direction of motion, the measurement of the lengths of rockets 1 and 2 in *Figures 3.4* and *3.5* will be considered. It will be assumed that the lightning marks are at the ends of the rockets.

It was assumed in *Figure 3.4*, that the lightning strikes simultaneously in the co-ordinate system Σ in which rocket 1 is at rest, such that the light signals reach C at the same time as shown in

Figure 3.4(c). An observer in rocket 1 will conclude that A coincided with A' when B coincided with B', so that, relative to rocket 1 (Σ), the *measured* lengths of the two rockets are the *same*. It was pointed out in Section 3.6 that according to the Lorentz transformations $(l_0)_2$ the proper length of rocket 2 is greater than $(l_0)_1$ the proper length of rocket 1. The measured lengths of the rockets are the same relative to Σ (*Figure 3.4(a)*), since rocket 2 is Lorentz contracted, whereas rocket 1 is not. In Section 3.6, we found

$$(l_0)_1 = (l_0)_2\sqrt{1 - v^2/c^2} = (l_0)_2/\gamma \tag{3.20}$$

It was shown in Section 3.6 that, if the light signals reach C simultaneously, as shown in *Figure 3.4(c)*, then the observer on rocket 2 would conclude that A' coincided with A before B' coincided with B, since the light from A' reaches C' (at a time t') before the light from B' (which arrives at C' at a time $t' + \Delta t'$). Consider the co-ordinate system Σ' in which rocket 2 is at rest. In this co-ordinate system, rocket 1 moves with velocity $-v$ as shown in *Figure 3.5*. The observer at rest on rocket 2 would record A' coinciding with A at a time $t' - A'C'/c$, as shown in *Figure 3.5(a)*, and would record B' coinciding with the mark B on rocket 1 at a later time $t' + \Delta t' - A'C'/c$. During the time interval between the events at A' and B' in Σ', according to the observer on rocket 2, A would have moved relative to A' such that at the time $t' + \Delta t' - A'C'/c$, the distance between the mark A on rocket 1 and mark B on rocket 1 is less than $A'B'$ (measured in Σ') as shown in *Figure 3.5(b)*. Thus the observer on rocket 2 would conclude that in Σ', the *measured* length of the moving rocket, number 1, is less than the length of the stationary rocket, number 2 in this instance. On the other hand, it was shown that an observer on rocket 1 would conclude that, relative to Σ, the measured lengths of the rockets are the same, as shown in *Figure 3.4(a)*. It can be seen that the different measures of length are intimately connected with the lack of absolute simultaneity. The effects were exaggerated by making the relative velocity of the rockets comparable with the speed of light.

The Lorentz transformations will now be used to fill in the mathematical background to the above qualitative discussion. In Σ' rocket 2 is at rest as shown in *Figure 3.5*, and its measured length $A'B'$ is its proper length $(l_0)_2 = x_1' - x_2'$. Relative to Σ', rocket 1, which has a proper length $(l_0)_1$, is moving with uniform velocity $-v$ and, according to the Lorentz transformations, should be Lorentz contracted such that its measured length l_1' relative to Σ' is

$$l_1' = (l_0)_1\sqrt{1 - v^2/c^2}$$

Using eqn (3.20)

$$l'_1 = (l_0)_2(1 - v^2/c^2) \tag{3.21}$$

As in Section 3.6, let the event at A, A' be at (x_1, t_1) and (x'_1, t'_1) relative to Σ and Σ' respectively, and let the event at B, B' be at (x_2, t_2) and (x'_2, t'_2) relative to Σ and Σ' respectively, where $t_1 = t_2$. Since $t_1 = t_2$ and $x_1 - x_2 = (l_0)_1$, from the Lorentz transformations, we have

$$t'_2 - t'_1 = \gamma v(x_1 - x_2)/c^2 = \gamma v(l_0)_1/c^2$$

Using eqn (3.20)

$$t'_2 - t'_1 = v(l_0)_2/c^2$$

Relative to Σ', rocket 1 moves with uniform velocity $-v$ for the time interval $(t'_2 - t'_1)$ between the events at A, A' and B, B'. Relative to Σ', *Figure 3.5*, the *measured* length of rocket 1 should be equal to $A'B'$ minus the distance travelled by the rocket in the time interval $(t'_2 - t'_1)$, that is

$$l'_1 = (l_0)_2 - v(t'_2 - t'_1) = (l_0)_2 - v^2(l_0)_2/c^2$$

This is in agreement with eqn (3.21). It can be seen that the Lorentz transformations fill in the mathematical background to the qualitative discussions given in Sections 3.6 and 3.7 and give a consistent account of the measurement of the times of the events and the lengths of the rockets relative to both Σ and Σ'.

The measurement of lengths perpendicular to the direction of relative motion will now be considered, using the example of the rockets. Let the lightning now strike at points A and A' and B and B' at the edges of the rockets, when the rockets are directly above each other, as shown in *Figure 3.6(a)*, leaving marks on both rockets. Let the observer on rocket 1 find that his light detectors at C were half way between the marks at A and B on rocket 1, and let the observer on rocket 2 find that his light detectors at C' were half way between the marks at A' and B' on rocket 2. Let AB be perpendicular to the direction of motion of rocket 2 as shown in *Figure 3.6.* Let the lightning strike simultaneously relative to rocket 1, such that the light signals reach the detectors at C at the same time as shown in *Figure 3.6(b)*. The observer on rocket 1 will conclude that A coincided with A' at the same instant as B coincided with B'. Now in the co-ordinate system Σ in which rocket 1 is at rest, C' moves in a direction perpendicular to AB so that, whatever the position of C' relative to rocket 1, $AC' = BC'$ as shown in *Figure 3.6(c)*. If space is isotropic and if the velocity of light is the same in

all directions, as is assumed in the theory of special relativity, then the light from A and B should reach C' at the same time relative to rocket 1. The coincidence of the light from both A and B with the detectors at C' is an event independent of any particular reference frame, so that the light from A and A' and B and B' must be

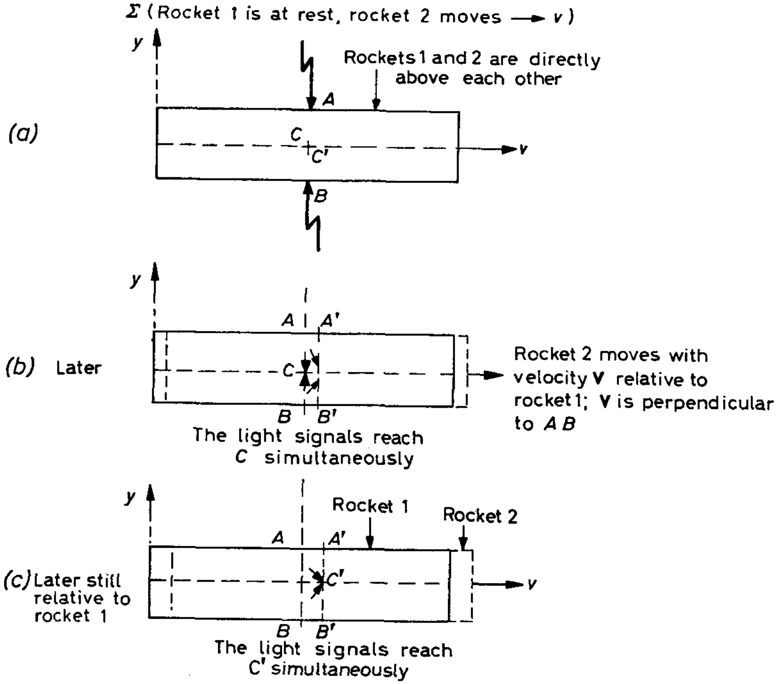

Figure 3.6. In this case the lightning strikes at the sides of rockets 1 *and* 2 *when they are directly above each other, such that relative to Σ, the co-ordinate system in which rocket* 1 *is at rest, the lightning strikes simultaneously, so that the light signals from A, A' and B, B' reach C, which is half way between A and B, simultaneously as shown in* (*b*). *In this case the light signals from A, A' and B, B' also reach C' which is half way between A' and B' on rocket* 2, *simultaneously as shown in* (*c*), *so that it is calculated that the lightning also strikes simultaneously relative to rocket* 2

measured to arrive at C' simultaneously relative to rocket 2. Since the observer on rocket 2 measures $A'C'$ to be equal to $B'C'$, he will deduce that A' coincided with A at the same instant as B' coincided with B. Thus the observers on both rockets 1 and 2 would agree, in this case, that A coincided with A' when B coincided with B'. If they chose the same units of length, they would agree that the length AB measured in Σ (rocket 1 at rest) was equal to the length

$A'B'$ measured in Σ' (rocket 2 at rest), or, in general,

$$y' = y$$
$$z' = z$$

in agreement with the Lorentz transformations.

It has been shown that, according to the theory of special relativity, the measured length of an object is reduced in the direction of relative motion when the object is moving relative to an observer measuring its length. This 'contraction' arises naturally if simultaneity is not absolute. On the old ether theories the Lorentz–FitzGerald contraction was attributed to motion of the body relative to the ether. According to these theories a body at *rest* on the earth should contract due to motion relative to the ether, which was considered an absolute frame of reference for electromagnetic phenomena. On the other hand, according to these theories, a body at rest relative to the ether, but moving relative to the earth, should not be contracted as seen from the earth. According to the theory of special relativity the change in measured length is associated with the *relative motion* between the object and the observer. The effect is perfectly reciprocal, and is due only to the relative motion between object and observer and not due to motion relative to any particular absolute reference frame. It is often asked whether the length contraction is 'real'. What the principle of relativity says is that the laws of physics are the same in all inertial frames, but the actual measures of particular quantities may be different in different co-ordinate systems. For example, in the example of a game of tennis on board a ship going out to sea, discussed in Section 1.4, it was reasonable within the context of Newtonian mechanics to find that the velocity of the tennis ball was different relative to the ship than relative to the beach. Is this change of velocity 'real'? According to the theory of special relativity, not only the measures of the velocity of the ball relative to the ship and the seashore will be different, but the measures of the dimensions of the tennis court parallel to the direction of relative motion and the measures of the time of events will also be different. Both the reference frames at rest relative to the ship and the beach can be used to describe the course of the game and the laws of physics will be the same in both systems, but the measures of certain quantities will differ.

In Newtonian mechanics it was possible to make statements such as: 'the length of the rod is 30 cm.' Since, in the theory of special relativity, the length of a body is not absolute, such a statement has

no meaning unless the length is specified relative to some standard of rest.

3.8. SYNCHRONIZATION OF SPATIALLY SEPARATED CLOCKS

In Section 3.6, simultaneity was only defined for a particularly simple case in which no actual measurement of time was involved. For purposes of discussion it was imagined previously that a series of clocks was distributed throughout space so that the time of events could be measured on the spot when the events happened. It was assumed that these clocks were good clocks and kept good time; but the question arises, how can such clocks be synchronized? One could suggest taking a standard clock around to each of them in turn and setting each clock in turn, but one cannot be sure that the standard clock would be unaffected by the transportation. It was pointed out in Section 1.3 that certain quantities have to be defined in terms of a particular theory, for example in Section 1.3 force and mass were defined in terms of Newton's laws of motion. In the theory of special relativity, in addition to force and mass, it is necessary to define how to synchronize spatially separated clocks in a way consistent with the theory of special relativity. The *definition* of distant simultaneity chosen by Einstein[1] was as follows:

> If at the point A of space there is a clock, an observer at A can determine the time values of events in the immediate proximity of A by finding the positions of the hands which are simultaneous with these events. If there is at the point B of space another clock in all respects resembling the one at A, it is possible for an observer at B to determine the time values of events in the immediate neighbourhood of B. But it is not possible without further assumption to compare, in respect of time, an event at A with an event at B. We have so far only defined an 'A time' and a 'B time'. We have not defined a common time for A and B. The latter time can now be defined in establishing *by definition* that the 'time' required by light to travel from A to B equals the time it requires to travel from B to A. Let a ray of light start at the 'A time' t_A from A towards B, let it at the 'B time' t_B be reflected at B in the direction of A, and arrive again at A at the 'A' time 't'_A'.
>
> In accordance with definition the two clocks synchronize if
>
> $$t_B - t_A = t'_A - t_B \tag{3.22}$$

According to the principle of the constancy of the velocity of light, the speed of light is the same in all directions of empty space, so that light should take the same time to go from A to B as from B to A. According to Einstein's definition, if the time recorded by

the two stationary clocks at A and B for the light to go from A to B is the same as the time they record for the light to go from B to A, then the clocks must be synchronous. If they were not synchronous, they would not measure the times to be the same. This definition of synchronization is based on the principle of the constancy of the velocity of light. If the velocity of light has the same numerical value in all regions of space, and if light travels in straight lines, then it is possible to synchronize all the clocks which are at rest in any inertial frame.

Equation (3.22) can be rewritten in the form

$$t_B = \tfrac{1}{2}(t_A + t'_A) \tag{3.23}$$

If an astronaut, in a space ship at rest relative to the inertial frame in which the earth is at rest, wanted to synchronize his watch with say Greenwich Mean Time, he could ask his base station on earth to send him a radio signal. On receipt of this signal, the astronaut should send a radio message back to earth immediately. He should also record the time on his watch when he received the radio signal. If the radio signal left earth at t_A G.M.T. and was received back on earth at t'_A G.M.T., then the time the astronaut received the radio signal was $\frac{1}{2}(t_A + t'_A)$ G.M.T. This information can be transmitted to the astronaut, who can then adjust his watch to read Greenwich Mean Time.

Alternatively, if he knew how far away he was from the earth, the astronaut could ask his base station to send him a message saying what the time was, for example, they could transmit a television picture of Big Ben. The astronaut would have to allow for the time the radio signal took to reach him. If he were on the moon, the signals would take $\sim 1\frac{1}{4}$ sec to reach him. If he were somewhere in the solar system the signals would take a time of the order of minutes to reach him. Even if one were on the earth and looked at a distant clock through a telescope or even at one's own wristwatch one should, in principle, allow for the time light takes to reach the eye from the clock. In normal everyday life, such a time delay is so small that it can generally be ignored, though, of course, in principle it is always present.

Einstein chose to define how to synchronize spatially separated clocks using light signals. This led many people to assume that the concept of time in the theory of special relativity depends on the law of propagation of light. Einstein's own view can be summarized by the following quotation[9]:

> The theory of relativity is often criticized for giving, without justification, a central theoretical role to the propagation of light, in that it founds

the concept of time upon the law of propagation of light. The situation is, however, somewhat as follows. In order to give physical significance to the concept of time, processes of some kind are required which enable relations to be established between different places. It is immaterial what kind of process one chooses for such a definition of time. It is advantageous, however, for the theory to choose only those processes concerning which we know something certain. This holds for the propagation of light *in vacuo* in a higher degree than for any other process which could be considered thanks to the investigations of Maxwell and H. A. Lorentz.

Any process other than light signals can be used to synchronize spatially separated clocks, such as elastic couplings or electrical signals transmitted along connecting wires. If the synchronization is carried out in a way consistent with the theory of special relativity, no contradiction with the light signals method should arise, if the theory of special relativity is correct. In practice, it is often simpler to use radar methods to determine the positions and times of distant events. An account of this method will now be given.

3.9. RADAR TECHNIQUES FOR MEASURING POSITIONS AND TIMES OF EVENTS

Radar techniques can be used to measure the times and positions of events, for example, to plot the position of a moving rocket. Let a directional antenna transmit a short radio pulse at a time t, as shown in *Figure 3.7*. This radio signal travels with a speed c in

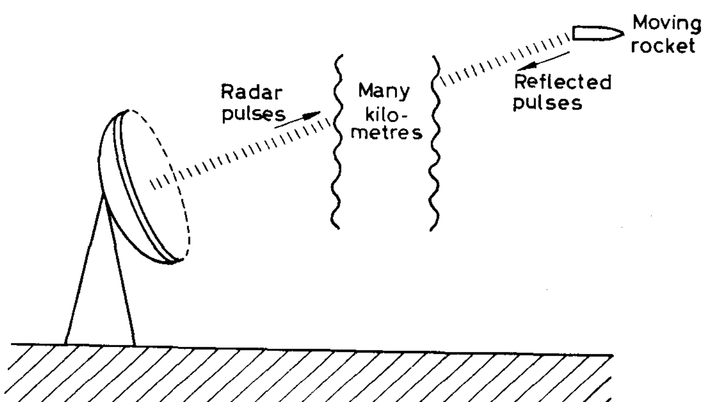

Figure 3.7. A directional antenna is used to transmit a radar pulse which is reflected by a moving rocket

empty space. When it reaches the rocket, a small fraction of the radio waves are reflected in the backward direction, in the direction of the transmitter. These signals can be received and amplified. Let the reflected signal be received back at the radar base at $t + \Delta t$. If the speed of radio signals is the same in both directions, as required by the principle of the constancy of the velocity of light, then it takes the radio signals the same time to reach the rocket as to return from the rocket. Hence the radio waves are reflected

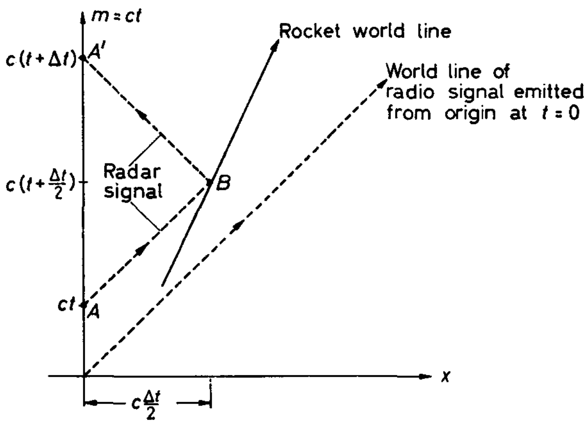

Figure 3.8. A space-time diagram showing the world line of the rocket. The radar pulse is transmitted from the radar base at the world point A having co-ordinates $x = 0$, $m = ct$. It is reflected by the rocket at the world point B at $x = c\,\Delta t/2$, $m = c(t + \Delta t/2)$ and received back at the radar base at the world point A′ at $x = 0$, $m = c(t + \Delta t)$

from the rocket at a time $t + \Delta t/2$. In a time interval $\Delta t/2$ the radio signals travel a distance $c\,\Delta t/2$. Hence at the time $t + \Delta t/2$, the rocket is at a distance of $c\,\Delta t/2$ from the radar base. If a directional antenna is used, the position of the rocket in space can be determined using only one clock, in the radar station on earth.

It is convenient to plot the successive positions of the rocket graphically. It will be assumed that the rocket moves along the x axis of an inertial frame Σ. It will be assumed that the radar station is at $x = 0$. It is conventional in the theory of special relativity to plot $m = ct$ against x as shown in *Figure 3.8*. Such a diagram is called a *space-time diagram*. The displacement of a rocket moving with uniform velocity is shown in *Figure 3.8*. Such a curve is called a 'world line'. Events are called 'world points'. The 'world lines' of light or radio signals, travelling with speed c

along the x axis, are at 45 degrees to the x and ct axes, as shown in *Figure 3.8*, since in a time Δt light signals travel a distance $\Delta x = c\Delta t$. The event corresponding to the emission of the radar pulse is represented by the world point A ($x = 0$, $m = ct$). The 'world line' of the radio signal is shown dotted. It reaches the rocket at the 'world point' B having co-ordinates $x = c\Delta t/2$; $m = c(t + \Delta t/2)$. The reflected radio signal reaches the radar base at the 'world point' A' having co-ordinates $x = 0$, $m = c(t + \Delta t)$. By repeating the experiment, the successive positions of the rocket can be determined and its velocity measured. If the rocket is moving with uniform velocity its 'world line' is straight as shown in *Figure 3.8.* If the rocket is accelerating, its 'world line' is curved.

Some readers may find it simpler to visualize the measurement of the times and co-ordinates of distant events, using the radar methods described above, rather than use all the paraphernalia of rulers and synchronized clocks distributed throughout space, with observers distributed throughout space to record events when and where they occur. Radar methods can be used to develop the Lorentz transformations in the way described in Appendix 6, where references to other works will be found. This method is now known as the method of the K-calculus.

In this Section, a space time diagram has been used to represent the displacement of a rocket relative to one inertial frame only. The method will be extended in Chapter 6 in such a way that the same 'world lines' and 'world points' can be used to represent the displacement of the rocket relative to more than one inertial reference frame.

Example. A rocket is moving directly away from the earth with uniform velocity. A radar pulse is transmitted from the earth at 12.00 G.M.T. The pulse reflected from the rear of the rocket is received back at the radar base at 12.02 G.M.T. The pulse reflected from the front of the rocket is received back at the radar base 2 μsec later. A second radar pulse is transmitted at 12.04 G.M.T. and, after reflection at the rear end of the rocket, is received back at base at 12.18 G.M.T. Find the speed and proper length of the rocket.

As practice, the reader should draw an *accurate* space-time diagram showing the 'world lines' of the radar signals and the path of the rocket as shown in *Figure 3.9(a)*. The reader can then estimate the speed of the rocket graphically.

If it returns to the radar base at 12.02 G.M.T., the radar pulse transmitted at 12.00 G.M.T. reaches the rear of the rocket at 12.01 G.M.T. The distance travelled by the radio signal in the 60 sec

between 12.00 and 12.01 G.M.T. is $60c$, or $1{\cdot}8 \times 10^{10}$ m. Thus at 12.01 G.M.T. the rear of the rocket is $1{\cdot}8 \times 10^{10}$ m from the earth.

Similarly, the radar signal transmitted at 12.04 G.M.T. reaches the rear of the rocket at 12.11 G.M.T. when it is $7 \times 60c$ or $12{\cdot}6 \times 10^{10}$ m from the earth. In the 10×60 sec between 12.01

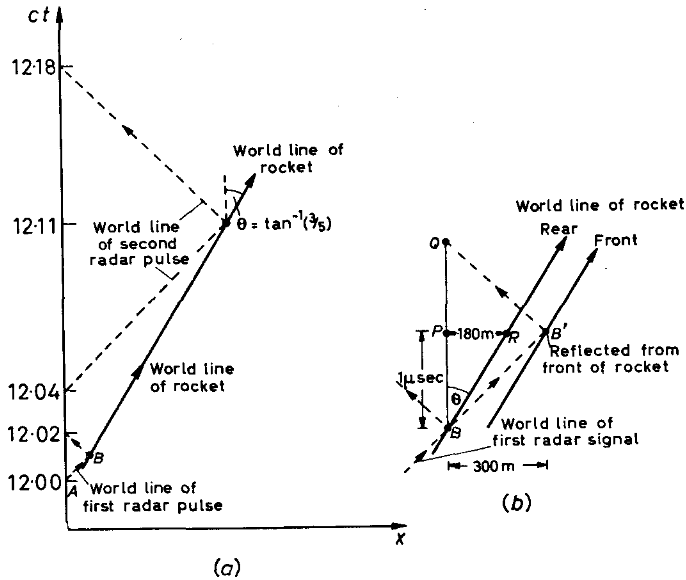

Figure 3.9. (*a*) *Space-time diagram showing the world lines of two radar signals sent to and reflected by the rear of the rocket. The world line of the rocket can be determined graphically. Since* $\tan\theta = 3/5$, *the speed of the rocket is* $0{\cdot}6c$. (*b*) *Enlarged space-time diagram around the world point B in Figure 3.9*(*a*). *The first radar pulse is reflected from the rear of the rocket at the world point B and from the front of the rocket at the world point B′*, 1 μsec *later. Hence* $PB' = 10^{-6}c = 300$ m. *The length of the rocket is the distance between the world lines of the rear and front of the rocket at a fixed time. For example, at* $12{\cdot}01 + (1\ \mu\text{sec})$ *the length of the moving rocket is given by RB′. By drawing the diagram to scale, the reader will find that the length of the moving rocket is* 120 m. *By trigonometry,* $PR = 300 \tan\theta = 180$ m, *so that* $RB' = 120$ m.

and 12.11 G.M.T. the rocket travels a distance of $6 \times 60c$. Hence the speed of the rocket is $0{\cdot}6c$.

The reader should draw the world lines of the front and rear of the rocket and the world lines of the first radar pulse on an enlarged scale and determine graphically the length of the moving rocket, as shown in *Figure 3.9*(*b*).

The first radar signal is reflected from the rear of the rocket at 12.01 G.M.T. and from the front of the rocket 1 μsec later. If v

is the speed of the rocket, in 1 μsec the front of the rocket moves a distance $10^{-6}v$, so that after passing the rear of the rocket, the radar pulse covers a distance $l + 10^{-6}v$ before reflection at the front end of the rocket 1 μsec later, where l is the length of the *moving* rocket. In 1 μsec the radar signal travels a distance $10^{-6}c$. Hence

$$10^{-6}c = l + 10^{-6}v$$

Since $v = 0{\cdot}6c$, we have

$$l = 10^{-6} \times 3 \times 10^{8}(1 - 0{\cdot}6) = 120\,\text{m}$$

This is the length of the moving rocket. From eqn (3.18) for the Lorentz contraction,

$$l = l_0\sqrt{1 - v^2/c^2}$$

Hence

$$l_0 = l/\sqrt{1 - v^2/c^2} = 120/\sqrt{1 - 0{\cdot}6^2}$$

or

$$l_0 = 150\,\text{m}$$

Hence the proper length of the rocket is 150 m. This example illustrates how one could determine the length of a moving body using radar techniques.

3.10. EXAMPLES OF TIME DILATION

3.10.1. *Gedanken Experimente*

Let a lamp, which is stationary in the inertial frame Σ', emit a pulse of light which travels a distance y_0' along the y' axis before it is reflected by a mirror, as shown in *Figure 3.10(b)*. In Σ' the time interval $\Delta t'$ between the light leaving the x' axis and returning

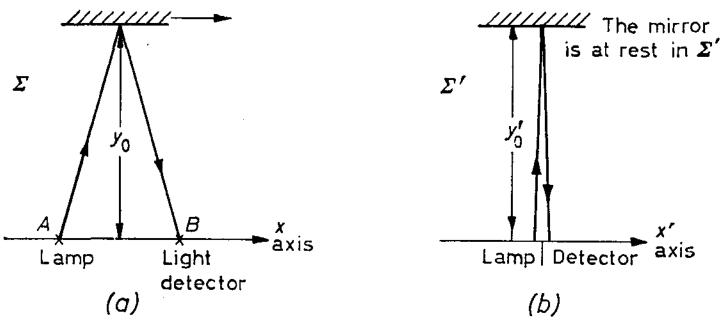

Figure 3.10. The measurement of a time interval in terms of the distance light travels in the time interval

is given by

$$\Delta t' = \frac{2y_0'}{c}$$

The motion of the pulse of light relative to Σ is illustrated in *Figure 3.10(a)*. Notice the light leaves at one point on the x axis and returns at another. The time interval Δt between the light's leaving and returning to the x axis is now given by

$$\Delta t = \frac{2\sqrt{y_0^2 + \left(\frac{v\Delta t}{2}\right)^2}}{c} \tag{3.24}$$

The same value is used for the velocity of light in Σ and Σ' so as to be in accord with the principle of the constancy of the velocity of light. Rearranging eqn (3.24), one has

$$\Delta t = \frac{2y_0}{c\sqrt{1 - v^2/c^2}}$$

In Section 3.7 it was shown directly from the postulates of the theory of special relativity that y_0 should be equal to y_0'. Hence, using the distance light travels as a measure of a time interval, and assuming c is the same in all inertial frames,

$$\Delta t = \Delta t'\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} \tag{3.25}$$

This is the expression for time dilation.

Notice that if the time interval between the time of emission of the light and the time of its return to the x axis were measured by clocks at rest in the two inertial frames, then one would need one such clock at rest in Σ' to measure the time interval $\Delta t'$ which would be a proper time interval. On the other hand, in Σ one would require two such clocks, one at rest at A to record the time the pulse left the x axis, and the other at rest at B to record the time of the return of the light pulse to the x axis. If the clocks at rest in Σ were synchronized as described in Section 3.8, then if the theory of special relativity is correct, the difference between the readings of the one clock in Σ' should be related by eqn (3.25) to the difference between the readings of the two clocks at rest at A and B in Σ, if the same unit of time is used in Σ and Σ'.

The relation (3.25) is perfectly reciprocal, since if the light returned at the same point in Σ, one would have $\Delta t = 2y_0/c$,

whereas the light pulse would return at different points on the x' axis in Σ', and one would then have

$$\Delta t' = \frac{2y_0'}{c\sqrt{1 - v^2/c^2}} \tag{3.26}$$

Experiments of the type described above have not been performed on the same system simultaneously to measure the *same* time interval in two different inertial frames, so that the relations (3.25) and (3.26) have not yet been confirmed directly by experiment. However, the time variable appears in the interpretation of many phenomena and the interpretations of some of these depend almost entirely on the Lorentz time transformations. Two experiments of this type, namely the Doppler effect and the decay of cosmic ray μ-mesons in the atmosphere, are now being discussed. Further examples based on the application of the Mössbauer effect are given in Section 8.4.

3.10.2. *The Doppler Effect*

The possibility that the motion of a source of light may affect the position of a line in a spectrum was pointed out by Doppler in 1842. At present, a simplified treatment of the Doppler effect is given; the general case is treated in Section 4.4. The present treatment is designed to try and bring out the role of the Lorentz time transformation. Let a source of light, which is at rest at O' the origin of Σ', emit light of frequency ν' when measured in Σ'. The emission of each maximum of the wavefront is treated as an event. Let the observer be at rest at O the origin of Σ. If Σ' moves with uniform velocity v relative to Σ, then, after the origins coincide, the source of light is moving with uniform velocity v away from the observer at rest in Σ. If a wave crest is emitted from O' at a time t' measured in Σ', then according to the Lorentz transformations, relative to Σ, this event would be recorded as happening at a position

$$x = \gamma(x' + vt') = \gamma vt'$$

at a time

$$t = \gamma(t' + vx'/c^2) = \gamma t'$$

This light then travels with a velocity c *in vacuo* from O' to O, and reaches O at a time x/c later. Hence, the wave crest *reaches* O at a time

$$T = t + \frac{x}{c} = \gamma t' + \gamma\frac{vt'}{c} = \gamma t'\left(1 + \frac{v}{c}\right)$$

Let the next maximum be emitted from O' at a time $t' + \frac{1}{\nu'}$, measured in Σ'. This maximum reaches O at a time

$$T + \Delta T = \gamma\left(t' + \frac{1}{\nu'}\right)\left(1 + \frac{v}{c}\right)$$

Hence

$$\Delta T = \frac{1}{\nu} = \frac{\gamma}{\nu'}\left(1 + \frac{v}{c}\right)$$

where ν is the frequency of the light reaching O measured in Σ.

$$\nu = \frac{\nu'\sqrt{(1 - v^2/c^2)}}{(1 + v/c)} \tag{3.27}$$

$$= \nu'\sqrt{\left(\frac{1 - v/c}{1 + v/c}\right)} = \nu'\sqrt{\left(\frac{c - v}{c + v}\right)} \tag{3.28}$$

Expanding to first order of v/c, one has

$$\nu = \nu'(1 - \tfrac{1}{2}v/c)(1 - \tfrac{1}{2}v/c) = \nu'(1 - v/c) \tag{3.29}$$

Hence, when the source is receding from the observer the frequency of the light should go down and the spectral line should be displaced towards the red end of the spectrum. If $c = \lambda\nu$ in Σ, and $c = \lambda'\nu'$ in Σ', then eqn (3.27) can be rewritten as

$$\lambda = \frac{\lambda'(1 + v/c)}{\sqrt{1 - v^2/c^2}} \tag{3.30}$$

To first order,

$$\lambda = \lambda'(1 + v/c) \tag{3.31}$$

When the source is approaching the observer,

$$\nu = \frac{\nu'\sqrt{1 - v^2/c^2}}{(1 - v/c)} = \nu'\sqrt{\frac{c + v}{c - v}} \simeq \nu'(1 + v/c) \text{ to first order} \tag{3.32}$$

and

$$\lambda = \frac{\lambda'(1 - v/c)}{\sqrt{1 - v^2/c^2}} \simeq \lambda'(1 - v/c) \text{ to first order} \tag{3.33}$$

Ives and Stilwell[10] measured accurately the apparent frequency of the blue-green H_β line of wavelength 4861 Å emitted by hydrogen atoms moving at high velocities in canal rays. The positive particles were produced in a hydrogen arc and were accelerated in an electric field. The spectrum of the light emitted in the direction of motion

of the canal rays was viewed directly by a spectrograph. Using a mirror the light emitted in the direction at 180 degrees to the direction of motion of the canal rays was reflected back on to the slit of the spectrograph. A typical photograph obtained by Ives and Stilwell is shown in *Figure 3.11(a)*. The photograph consists of a

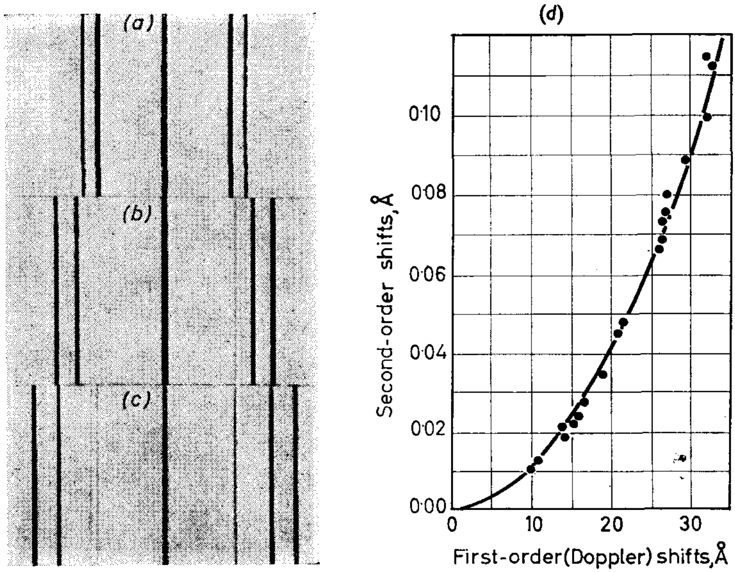

Figure 3.11. (a), (b) and (c) Typical spectra obtained by Ives and Stilwell[10]*. The spectra consist of central undisplaced lines with 'symmetrically' placed lines on either side; the accelerating potential differences were (a)* 7859, *(b)* 13702, *and (c)* 20755 V *respectively; (d) the second-order shift* $\lambda_m - \lambda' = \left(\frac{1}{2}\frac{v^2}{c^2}\lambda'\right)$ *is plotted against the first-order Doppler shift* $\Delta\lambda = (v/c)\lambda'$ *for various accelerating potentials. It can be seen that the experimental points are in very good agreement with the predictions of the theory of special relativity, which are shown by the continuous line.* (By courtesy of the Editor, *Sci. Proc. Roy. Dublin Soc*[11].)

central line with symmetrically placed lines on either side. There were always some stationary atoms present and they gave rise to the central line. On either side of the central line are the displaced lines; the lines due to canal rays coming towards the spectrograph are displaced towards the blue, and those due to the reflected light, emitted by the same canal rays in a direction directly away from the slit, are displaced towards the red end of the spectrum. In *Figure 3.11(a)* there are two lines on either side of the central line. These correspond to molecular aggregates of hydrogen of double

and triple mass respectively. To first order of v/c both the classical and relativistic theories predict a shift of $\Delta\lambda = \pm \frac{v}{c}\lambda'$, where λ' is the wavelength of the light in the inertial frame in which the source of light is at rest. The displacement $\Delta\lambda$ enabled Ives and Stilwell to calculate v/c in each case. Ives and Stilwell calculated the average wavelength λ_m of the displaced lines. According to the theory of special relativity, from eqns (3.30) and (3.33) one has

$$\lambda_m = \frac{\lambda_1 + \lambda_2}{2} = \frac{\gamma}{2}[\lambda'(1 - v/c) + \lambda'(1 + v/c)]$$

$$= \frac{\lambda'}{\sqrt{1 - v^2/c^2}} \qquad (3.34)$$

therefore
$$\lambda_m - \lambda' = \lambda'\left[\frac{1}{\sqrt{1 - v^2/c^2}} - 1\right] \simeq \frac{v^2}{2c^2}\lambda' \qquad (3.35)$$

Typical sets of results obtained by Ives and Stilwell are shown in Table 3.1 and in *Figure 3.11*(*d*). It will be seen that for all accelerating potentials the results are in good agreement with the predictions

Table 3.1

Accelerating potential difference (V)	$\Delta\lambda = \frac{v}{c}\lambda'$ (Å)	$\frac{1}{2}\frac{v^2}{c^2}\lambda'$ *calculated* (Å)	$\lambda_m - \lambda'$ *observed* (Å)
26,735	25·82	0·0670	0·067
34,395	29·40	0·0869	0·090
40,190	31·93	0·1049	0·0995
42,260	32·50	0·1098	0·113

of the theory of special relativity. For second order terms, the predictions of the classical theory depend to some extent on the velocity of the ether wind, but whatever value is chosen for the ether wind the predictions differ from eqn (3.35) in terms of the second order. Thus the relativistic theory of the Doppler effect, which is based so intimately on the Lorentz time transformation, was confirmed by Ives and Stilwell to terms of second order of v/c.

3.10.3. *The Decay of Cosmic Ray μ-mesons in the Atmosphere*

According to the law of radioactive decay, if a number N_0 of radioactive atoms are at rest at the origin of an inertial frame Σ' at $t' = 0$, then after a time t' the number remaining is given by

$$N = N_0 e^{-t'/T_0} \qquad (3.36)$$

where T_0 is the average time a radioactive atom lives before it decays when it is at rest. The radioactive atoms are moving along the x axis relative to the inertial frame Σ, so that according to the Lorentz transformations corresponding to a point $x' = 0$ at a time t' in Σ', one has in Σ, $x = \gamma vt'$ and $t = \gamma t'$. Substituting in eqn (3.36) one obtains

$$N = N_0 e^{-t/\gamma T_0} \tag{3.37}$$

Hence, relative to Σ, the radioactive atoms should be measured to live an average time of γT_0 and not T_0, so that the average lifetime of radioactive atoms should be measured to be longer when the radioactive atoms are moving relative to the laboratory. No experiment has so far been performed with radioactive atoms. A variety of fundamental particles have been discovered having masses between the mass of the electron and the mass of the proton. These undergo spontaneous decay and have short lifetimes. The first to be discovered was the μ-meson, which was first observed in the cosmic radiation. The average lifetime of μ-mesons when they are at rest has been shown experimentally to be $T_0 = 2{\cdot}2 \times 10^{-6}$ sec. (Rosser[12], Section 3.8.2.)

Now the μ-mesons in the cosmic radiation are generally produced near the top of the atmosphere as a result of the nuclear interactions produced by the primary cosmic radiation. In these interactions π-mesons are produced, which subsequently decay into μ-mesons. Now, if μ-mesons only lived for a time of $T_0 = 2{\cdot}2 \times 10^{-6}$ sec relative to the earth, even if they travelled with the velocity of light, they would only travel an average distance of $cT_0 = 3 \times 10^8 \times 2{\cdot}2 \times 10^{-6}$ m $\simeq$ 660 m before decaying. If this were correct, virtually none of the μ-mesons produced near the top of the atmosphere would be expected to reach sea level, whilst in fact a substantial proportion of the μ-mesons do. Hence, the μ-mesons reaching sea level must either have velocities exceeding the velocity of light or must live longer than $2{\cdot}2 \times 10^{-6}$ sec relative to the earth. No velocities exceeding the velocity of light have been recorded and so the phenomenon has been interpreted as an increase of the apparent lifetime of the μ-mesons by a factor $\gamma = 1/(1 - v^2/c^2)^{\frac{1}{2}}$, where v is the velocity of the μ-mesons relative to the earth. This is in agreement with the predictions of the theory of special relativity. The μ-mesons at sea level have momenta $\sim 3 \times 10^9$ eV/c corresponding to $\gamma \sim 30$. According to eqn (3.37) the lifetime of a μ-meson of momentum 3×10^9 eV/c should be $\sim 30 \times 2{\cdot}2 \times 10^{-6}$ sec relative to the earth, and such a μ-meson should travel an average distance $\sim$20 km relative to the earth before decaying, and therefore should have a substantial probability of reaching sea level.

Hence, the behaviour of the μ-meson in the cosmic radiation can be accounted for satisfactorily in terms of the theory of special relativity. It is interesting to interpret the co-ordinates and times of the events of creation and decay of a μ-meson in terms of the Lorentz transformations. Normally, the lifetimes of μ-mesons are distributed statistically, but for purposes of discussion, it will be assumed that a μ-meson lives for a time T_0 in the inertial frame in which it remains at rest. Ionization loss will be neglected, so that the velocity of the μ-meson remains constant relative to the earth. Let the origins of Σ and Σ' coincide at $t = t' = 0$ at the point of creation of a

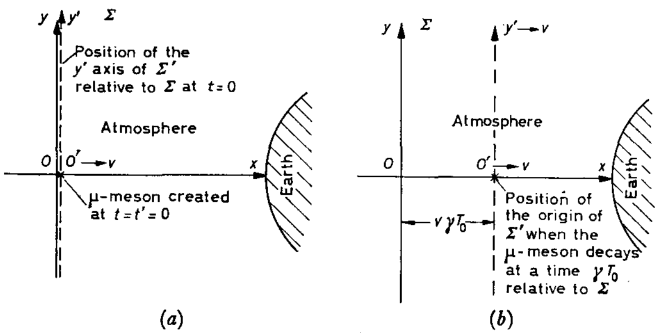

Figure 3.12. (a) A μ-meson is created at the origin of Σ' at the instant $t' = 0$ when the origins of Σ and Σ' coincide. If the μ-meson moves with uniform velocity v relative to Σ it remains at rest at the origin of Σ'. (b) The position of the μ-meson when it decays

μ-meson, as shown in *Figure 3.12(a)*. Let Σ' move with the same velocity v as the μ-meson so that the μ-meson remains at rest at O', the origin of Σ'. In Σ' the μ-meson would be created at $x' = 0$, $t' = 0$ and would decay at $x' = 0$, $t' = T_0$. In Σ it would be created at $x = 0$, $t = 0$ and would decay at

$$x = \gamma(x' + vt') = \gamma v T_0$$

$$t = \gamma(t' + vx'/c^2) = \gamma T_0$$

Hence, an observer in Σ (that is on the earth) would say that the μ-meson travelled with a velocity v for a time γT_0 covering a distance $\gamma v T_0$, so that relative to the earth we have time dilation. An observer at rest in Σ' would say that, relative to the μ-meson, the origin of Σ and also the earth had travelled with a velocity v for a time T_0 covering a distance vT_0, by the time the μ-meson was observed to decay in Σ'. This distance is less, by a factor γ, than

the distance travelled by the μ-meson relative to the earth. Relative to the μ-meson, there is no time dilation, but the earth and the earth's atmosphere are Lorentz contracted.

3.10.4. *The Decay of π-mesons in the Laboratory*

Beams of charged π-mesons of kinetic energies up to 6 GeV can now be produced in the laboratory by accelerators, such as the proton synchrotron at C.E.R.N. Charged π-mesons undergo spontaneous decay into μ-mesons and neutrinos. The mean lifetime of π-mesons, when they are at rest, is $2{\cdot}55 \times 10^{-8}$ sec. The rest mass of π-mesons is $139{\cdot}6$ MeV/c^2.

Problem—Calculate the average distance (mean free path) π-mesons of velocity (*a*) $0{\cdot}75c$, (*b*) $0{\cdot}9c$ (*c*) $0{\cdot}99c$ (*d*) $0{\cdot}999c$ (*e*) $0{\cdot}9995c$ (corresponding to kinetic energies of approximately (*a*) 72 MeV (*b*) 180 MeV (*c*) 850 MeV (*d*) 3 GeV (*e*) 4·4 GeV respectively) will travel before they undergo spontaneous decay. (The formula relating kinetic energy and velocity will be developed in Chapter 5. 1 GeV = 10^9 eV and 1 MeV = 10^6 eV. The electron volt is defined in Section 5.3.3.)

From eqn (3.37) the mean lifetime of moving π-mesons relative to the laboratory is γT_0, where $T_0 = 2{\cdot}55 \times 10^{-8}$ sec and $\gamma = (1 - \beta^2)^{-\frac{1}{2}}$ where $\beta = u/c$. If u is their velocity, the *average* distance π-mesons will travel before undergoing spontaneous decay is $u\gamma T_0$. If β is $0{\cdot}75$, $\gamma = 1{\cdot}512$. Hence,

$$\gamma T_0 = 1{\cdot}512 \times 2{\cdot}55 \times 10^{-8}\text{ sec} = 3{\cdot}87 \times 10^{-8}\text{ sec}$$

Mean distance $= u\gamma T_0 = 0{\cdot}75c \times 3{\cdot}87 \times 10^{-8}\text{ sec} = 8{\cdot}7$ m.

The values for the other cases are

(*b*) $\beta = 0{\cdot}9$ $\quad \gamma = 2{\cdot}294$

$\gamma T_0 = 5{\cdot}84 \times 10^{-8}$ sec $\quad u\gamma T_0 = 15{\cdot}7$ m

(*c*) $\beta = 0{\cdot}99$ $\quad \gamma = 7{\cdot}089$

$\gamma T_0 = 18{\cdot}1 \times 10^{-8}$ sec $\quad u\gamma T_0 = 54$ m

(*d*) $\beta = 0{\cdot}999$ $\quad \gamma = 22{\cdot}37$

$\gamma T_0 = 57 \times 10^{-8}$ sec $\quad u\gamma T_0 = 170$ m

(*e*) $\beta = 0{\cdot}9995$ $\quad \gamma = 31{\cdot}63$

$\gamma T_0 = 81 \times 10^{-8}$ sec $\quad u\gamma T_0 = 244$ m

An experiment with π-mesons of velocity $\sim 0{\cdot}75c$ and kinetic energy $\sim$72 MeV was carried out by Durbin, Loar and Havens[13] in 1952. They found experimentally that the average distance such

π-mesons travelled before they decayed was $8{\cdot}5 \pm 0{\cdot}6$ m. If there were no time dilation, and if their velocity were $0{\cdot}75c$, the π-mesons would only travel an average distance of $5{\cdot}7$ m before decay.

More recently, experiments have been carried out with π-mesons of kinetic energies up to $\sim$6 GeV. The above problem shows that π-mesons of kinetic energy $4{\cdot}5$ GeV should travel an average distance of about 250 m, before undergoing spontaneous decay. Experiments have shown that the speed of light is a limiting speed for particles (cf. Section 5.4.2). Thus, but for time dilation, π-mesons could only travel a maximum average distance of $2{\cdot}55 \times 10^{-8}c$ or $7{\cdot}6$ m before decaying. In practice, high energy nuclear physicists can place their π-meson detectors, such as bubble chambers, at a distance of 100 m or more from the point where the π-mesons are produced, since, allowing for time dilation the mean free path for decay is about 250 m for $4{\cdot}5$ GeV π-mesons. But for time dilation the intensity of $4{\cdot}5$ GeV π-mesons would decrease to $\exp(-100/7{\cdot}6)$ or $\sim 2 \times 10^{-6}$ of its original intensity in 100 m. Since the speed of light has been shown to be the limiting speed for particles, the π-mesons must live longer than $2{\cdot}55 \times 10^{-8}$ sec relative to the laboratory, so that these experiments confirm time dilation. Relative to the reference frame in which the π-mesons are at rest, the dimensions of the laboratory are Lorentz contracted in a direction parallel to the direction of relative motion.

High energy nuclear physicists are now working in an energy range where the use of the equations of the theory of special relativity is imperative. To quote Kittel, Knight and Ruderman[14]:

> It has been said that almost every high-energy physicist tests special relativity every day. He uses the Lorentz transformation with the same confidence that physicists in the nineteenth century used Newton's laws.

It has been shown that the predictions of the theory of relativity agree with the experimental results in the case of the Doppler effect and in the case of μ-meson decay. This gives one confidence in the use of the time transformation. It must be pointed out that time appears as a variable in the interpretation of many other phenomena, for example, velocity is equal to the distance travelled per unit time. The successes of the relativistic velocity transformations in interpreting experiments are indirect evidence in favour of the time transformation. Momentum is defined as the product of mass times velocity. The development of relativistic dynamics in Chapter 5 is based on the Lorentz transformations, and the successes of this branch of the theory of special relativity are again indirect evidence in favour of the time transformation.

It must be emphasized that in this section it was assumed that Σ' always moved with the same uniform velocity v relative to Σ. Under these conditions, the two inertial frames were perfectly reciprocal. The discussion of what may or may not happen when a clock is accelerated, such that it undergoes a journey in a closed path is deferred until Chapter 8. Such a journey necessarily involves the acceleration of the clock from one inertial reference frame to another at some stage during the journey.

Example. The furthest star in our galaxy is about 10^5 light years away. Explain in terms of time dilation (or length contraction) how it is possible, *in principle*, for a human being to reach this star within his normal life span. Estimate what uniform velocity a rocket would require to reach the star in 10 years, measured in the inertial reference frame in which the rocket is at rest.

Let the earth be at the origin of an inertial frame Σ, and let the rocket be at the origin of Σ'. Let the rocket leave the earth at $t = t' = 0$ and travel with velocity v, $[\gamma = (1 - v^2/c^2)^{-\frac{1}{2}}]$ for a time $t' = 10k$ relative to the rocket, where k is the number of seconds in a year. Let the rocket reach the star at a distance of 10^5 light years or 10^5kc metres from the earth at a time t relative to Σ. From the Lorentz transformations, since $x' = 0$, we have

$$t = \gamma(t' + vx'/c^2) = \gamma t' = \gamma 10k$$

This is the normal expression for time dilation. The time for the journey relative to Σ' can be measured by one clock on the rocket and so is a proper time interval. The time for the journey relative to Σ must be measured by two spatially separated synchronized clocks at rest relative to the earth. If v is the speed of the rocket the distance travelled by the rocket relative to the earth in a time $t = \gamma 10k$ sec is $\gamma 10kv$. But the distance from the earth to the star is 10^5 light years or 10^5kc metres. Hence

$$10^5kc = \gamma 10kv$$

{Alternatively, one can use the Lorentz transformation

$$x = \gamma(x' + vt')$$

where $x = 10^5kc$, $x' = 0$ and $t' = 10k$}. Hence

$$10^4c = v/\sqrt{1 - v^2/c^2}$$

$$10^8(1 - v^2/c^2) = v^2/c^2$$

or

$$v^2/c^2 = 10^8/(1 + 10^8) = 1/(1 + 10^{-8})$$

or

$$v \simeq (1 - 5 \times 10^{-9})c \tag{3.38}$$

and

$$\gamma \simeq 10^4$$

Thus due to time dilation, the measured time for the journey is less relative to the rocket than relative to the earth, so that, *in principle*, it is possible for an astronaut to reach the star in his normal life span. However, it will be shown in Section 5.10 that, due to fuel considerations, it is impossible to accelerate conventional rockets to such high speeds, so that, at present at least, the question is of academic interest only. However, the protons in the cosmic radiation do have such high speeds. For example, a proton of velocity given by eqn (3.38) would have an energy of $\sim 10^{13}$ eV. Protons of energies up to $\sim 10^{19}$ eV have been observed in the cosmic radiation. Protons of energy 10^{19} eV have a value of $\gamma \sim 10^{10}$. If they were not deflected by galactic magnetic fields, such protons would reach the edge of the galaxy in about 5 min, measured in the rest frame of the proton.

Relative to the rocket, the galaxy moves with a velocity $-v$, where v is given by eqn (3.38), for a time of 10 years covering a total distance of $10kv \simeq 10kc$ relative to the rocket. Relative to the earth, the distance from the earth to the star is 10^5kc. Hence, relative to the moving rocket, the distance from the earth to the star is Lorentz contracted by a factor $\gamma \sim 10^4$. Relative to a proton of energy 10^{19} eV the dimensions of the galaxy are Lorentz contracted by a factor $\gamma \sim 10^{10}$ and, in the direction of relative motion of the proton and the galaxy, relative to the proton the galaxy would have roughly the dimensions of the solar system.

3.11. THE PROPAGATION OF SPHERICAL WAVES IN EMPTY SPACE

Consider a spherical array of photomultipliers at rest in empty space at a distance r from the origin of the inertial frame Σ. If light is emitted from the origin of Σ at a time $t = 0$, then, according to the principle of the constancy of the velocity of light, the light should reach all the photomultipliers at rest in Σ at the same time $t = r/c$ relative to Σ. Similarly, if there is a spherical array of photomultipliers at rest in Σ' at a distance r' from the origin of Σ', then, according to the principle of the constancy of the velocity of light, the light emitted from the origin of Σ' at the time $t' = 0$, when the origins of Σ and Σ' coincide, should reach all the photomultipliers

at rest in Σ' at a time $t' = r'/c$ measured in Σ'. According to the principle of the constancy of the velocity of light, observers at rest in Σ and Σ' respectively should both record that the light was travelling outwards as a spherical wave in their respective co-ordinate systems, and with their respective origins as centres of the wavefront. It will now be shown how these two, apparently contradictory, viewpoints can be reconciled using the Lorentz transformations.

Consider two photomultipliers at rest at points A and B on the x' axis of Σ' at distances $+r'$ and $-r'$ respectively from the origin

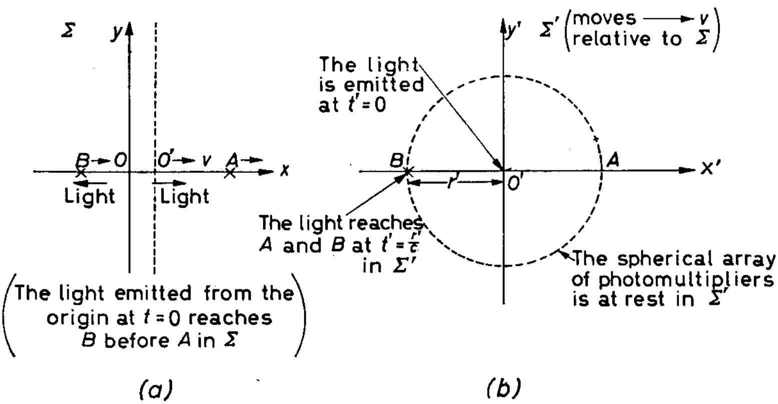

Figure 3.13. The two photomultipliers A and B are at rest in Σ' as shown in (b). Light emitted from the origins of Σ and Σ' at $t = t' = 0$ reaches A and B simultaneously relative to Σ'. The photomultipliers A and B are moving relative to Σ; B moves towards O so that light emitted from O at $t = 0$ reaches B before it reaches A, relative to Σ

O', as shown in *Figure 3.13*(*b*). According to the principle of the constancy of the velocity of light, the light emitted from O' at the time $t' = 0$ should reach the photomultipliers at A and B simultaneously at a time $t' = r'/c$ in Σ'. However, an observer at rest in Σ would say that, initially, the photomultiplier at B was moving towards the light, whilst the photomultiplier at A was moving away from the light as shown in *Figure 3.13*(*a*). Relative to Σ the light should be detected by the photomultiplier at B before the light reaches A. According to the Lorentz transformations the observer in Σ would record the detection of the light at B at co-ordinates and time given by

$$x_B = \gamma(-r' + vt'); \quad t_B = \gamma(t' - vr'/c^2)$$

and corresponding to the detection of the light by the photomultiplier at A, he should record

$$x_A = \gamma(r' + vt'); \quad t_A = \gamma(t' + vr'/c^2)$$

Thus, according to the Lorentz transformations, the observer at rest in Σ would record the light reaching B before A, but he would also record that the distance the light travelled to B was less than to A. One has

$$\frac{x_A}{t_A} = \frac{\gamma[r' + vt']}{\gamma[t' + vr'/c^2]} = \frac{[r'/t' + v]}{[1 + (v/c^2)(r'/t')]}$$

Now, if $r'/t' = c$, then

$$\frac{x_A}{t_A} = \frac{c + v}{1 + v/c} = c$$

Similarly,

$$x_B/t_B = c$$

A similar argument can be applied in the inertial frame Σ' to the detection of the light by the photomultipliers at rest in the inertial frame Σ. Due to the different measures of *both* the spatial and temporal separation of the events of emission and detection of the light, the observers at rest in Σ and Σ' can both obtain the same numerical value for the velocity of light. They can both record the propagation of the light as a spherical wave with their respective origins as the centres of the wavefronts, though when one of the observers detects the wavefront with his 'stationary' array of photomultipliers the other observer would not record these events as simultaneous, nor on the surface of a sphere in his own frame of reference.

3.12. INTERVALS BETWEEN EVENTS

Let an event occur at the point x, y, z at a time t in the inertial frame Σ. Let another event occur at the point $x + \delta x$, $y + \delta y$, $z + \delta z$ at a time $t + \delta t$. In the inertial frame Σ' let these events occur at x', y', z' and $x' + \delta x'$, $y' + \delta y'$, $z' + \delta z'$ at times t' and $t' + \delta t'$ respectively. According to the Galilean transformations one would have

$$\delta x = \delta x' + v\,\delta t'$$

$$\delta y = \delta y'$$

$$\delta z = \delta z'$$

$$\delta t = \delta t'$$

According to the Galilean transformations the temporal interval between the two events should be an invariant. If $\delta t'$ and δt were zero,

$$\delta x^2 + \delta y^2 + \delta z^2 = \delta x'^2 + \delta y'^2 + \delta z'^2$$

which means that the distance between two points measured at the same time in Σ and Σ' would be an invariant; that is, lengths are absolute according to the Galilean transformations.

According to the Lorentz transformations one has

$$\begin{aligned} \delta x &= \gamma[\delta x' + v\,\delta t'] \\ \delta y &= \delta y' \\ \delta z &= \delta z' \\ \delta t &= \gamma[\delta t' + v\,\delta x'/c^2] \end{aligned}$$

According to the Lorentz transformations the temporal interval between two events is not an invariant. Now

$$\delta x^2 + \delta y^2 + \delta z^2 = \gamma^2(\delta x'^2 + 2v\,\delta x' \,.\, \delta t' + v^2\,\delta t'^2) + \delta y'^2 + \delta z'^2$$

so that, even if $\delta t'$ were zero, $\delta x^2 + \delta y^2 + \delta z^2$ would not be equal to $\delta x'^2 + \delta y'^2 + \delta z'^2$. However,

$$\begin{aligned} &\delta x^2 + \delta y^2 + \delta z^2 - c^2\,\delta t^2 \\ &\quad = \gamma^2[\delta x' + v\,\delta t']^2 + \delta y'^2 + \delta z'^2 - c^2\gamma^2\left[\delta t' + \frac{v\,\delta x'}{c^2}\right]^2 \\ &\quad = \gamma^2[\delta x'^2 + 2v\,\delta x'\,\delta t' + v^2\,\delta t'^2 - c^2\,\delta t'^2 - 2v\,\delta x'\,\delta t' \\ &\qquad\quad - v^2\,\delta x'^2/c^2] + \delta y'^2 + \delta z'^2 \\ &\quad = \gamma^2[\delta x'^2(1 - v^2/c^2) - c^2\,\delta t'^2(1 - v^2/c^2)] + \delta y'^2 + \delta z'^2 \\ &\quad = \delta x'^2 + \delta y'^2 + \delta z'^2 - c^2\,\delta t'^2 \end{aligned}$$

so that

$$\delta s^2 = \delta x^2 + \delta y^2 + \delta z^2 - c^2\,\delta t^2 \tag{3.39}$$

is an invariant in all inertial frames moving with uniform velocity relative to each other. The quantity δs is called *the* interval between the two events. [Some writers prefer to define δs^2 such that

$$\delta s^2 = c^2\,\delta t^2 - (\delta x^2 + \delta y^2 + \delta z^2)$$

We shall *always* define δs^2 according to eqn (3.39).] According to the theory of special relativity if two events occur, two observers, one at rest in Σ and the other at rest in Σ', will record different distance and time separations between the two events, but they will

record the same interval δs. If δs^2 is an invariant, then δs^2 cannot change sign when one transforms from one inertial frame to the other. If δs^2 is positive, as defined by eqn (3.39), then the interval δs between the two events is called a *space-like* interval, and, if δs^2 is negative, δs is called a *time-like* interval. The space-like and time-like properties of δs will be illustrated by a few numerical examples.

Let one event occur at $x = 0$ at $t = 0$, and let the other event occur at $x = 6c$, $t = 10$ sec relative to the inertial frame Σ. It is assumed for the remainder of this section that y and z are always zero for all the events. (The expression $6c$ represents $6 \times 3 \times 10^8$ m and is assumed to have the dimensions of a length.) In Σ we have

$$\delta s^2 = \delta x^2 - c^2\,\delta t^2 = (6c)^2 - c^2 10^2 = -64c^2$$

so that δs is a *time-like* interval.

Now consider an inertial frame Σ' moving with uniform velocity $c/2$ relative to Σ. In Σ' one has

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{2}{\sqrt{3}}$$

$$\delta x' = \gamma[\delta x - v\,\delta t] = \frac{2}{\sqrt{3}}\left[6c - \frac{c}{2}\,10\right] = \frac{2c}{\sqrt{3}}$$

$$\delta t' = \gamma\left[\delta t - \frac{v\,\delta x}{c^2}\right] = \frac{2}{\sqrt{3}}\left[10 - \frac{c6c}{2c^2}\right] = \frac{14}{\sqrt{3}}$$

Hence, in Σ',

$$\delta s^2 = \left(\frac{2c}{\sqrt{3}}\right)^2 - c^2\left(\frac{14}{\sqrt{3}}\right)^2 = -64c^2$$

illustrating that δs^2 is invariant.

Now let $v = 3c/5$, such that $\gamma = \frac{5}{4}$. In this case $\delta x' = 0$, $\delta t' = 8$ sec; $\delta s^2 = -64c^2$. Notice that in this case $\delta x'$ is zero, so that in this inertial frame the two events are recorded at the same place. In general, $\delta x'$ is zero in the equation

$$\delta x' = \gamma(\delta x - v\,\delta t)$$

when

$$v = \delta x/\delta t \qquad \text{or} \qquad \frac{v}{c} = \frac{\delta x}{c\,\delta t} \tag{3.40}$$

If δs is time-like, δs^2 is negative, $c^2\,\delta t^2 > \delta x^2$ and $c\,|\delta t| > |\delta x|$. Thus, if δs is time-like, it is always possible to satisfy eqn (3.40) with $v < c$ and find a reference frame in which $\delta x'$ is zero. In this

reference frame, the time interval between the two events is a proper time interval, since the two events are recorded at the same position. Let the proper time interval be denoted by $\delta\tau$. Equation (3.39) becomes

$$\delta s^2 = \delta x^2 + \delta y^2 + \delta z^2 - c^2\,\delta t^2 = -c^2\,\delta\tau^2$$

If v is given by eqn (3.40),

$$\delta t' = \gamma\left(\delta t - \frac{v\,\delta x}{c^2}\right) = \gamma\,\delta t\left(1 - \frac{\delta x^2}{c^2\,\delta t^2}\right)$$

Since δs^2 is negative, that is $c^2\,\delta t^2 > \delta x^2$, $\delta t'$ always has the same sign as δt, so that, when δs is time-like, the time order of the two events is the same in all inertial reference frames.

Consider an example in which the first event is again at $x = 0$, $t = 0$, but the second event is now at $x = 10c$, $t = 6$ sec in the inertial frame Σ. The interval δs^2 is now space-like and is equal to $+64c^2$. In the inertial frame Σ' having velocity $c/2$ relative to Σ, one now has $\delta x' = 14c/\sqrt{3}$ and $\delta t' = 2/\sqrt{3}$ such that the interval is again $64c^2$. In an inertial frame moving with velocity $v = \frac{3}{5}c$ relative to Σ, one has

$$\delta x' = 8c; \quad \delta t' = 0; \quad \delta s^2 = +64c^2$$

In this last reference frame, the events would be measured to happen at the same time. It is possible to choose such an inertial frame whenever δs^2 is positive and δs is a space-like interval.

In an inertial frame moving with velocity $v = \frac{4}{5}c$ relative to Σ, we have

$$\delta x' = \frac{26c}{3}, \quad \delta t' = -\frac{10}{3}\text{ sec}; \quad \delta s^2 = +64c^2$$

Thus in the inertial frame moving with velocity $\frac{4}{5}c$ relative to Σ, the 'second' event is recorded as happening before the 'first', so that the order of events is reversed.

In the equation

$$\delta t' = \gamma\left(\delta t - \frac{v\,\delta x}{c^2}\right) = \gamma\,\delta t\left(1 - \frac{v\,\delta x}{c^2\,\delta t}\right)$$

the time interval $\delta t'$ is zero if

$$\frac{v}{c} = c\,\frac{\delta t}{\delta x}$$

If δs is space-like, δs^2 is positive, $\delta x^2 > c^2\,\delta t^2$ and $|\delta x| > c\,|\delta t|$. Hence if δs is space-like, it is possible to find an inertial reference

frame in which $\delta t'$ is zero. If $v/c > c\,\delta t/\delta x$, but less than unity, then $\delta t'$ is opposite in sign to δt and the time order of the events is different in Σ and Σ', as illustrated in the above numerical example.

The reversal of the time sequence of events in different inertial frames is possible whenever δs is a space-like interval. This prediction of the theory of special relativity provoked a lot of discussion at one time. Before the introduction of the theory of relativity and the advent of quantum mechanics, it was believed that physics was strictly deterministic. It was believed that a given cause always gives rise to the same effect. Under the influence of the prevailing ideas of absolute time, it was believed that it was possible to have a universal definite time order for all events, and all earlier events could in principle affect all later events. It seemed absurd that the order of some events could be different in different inertial frames. Now for events for which δs^2 is positive, one cannot send a light signal from one event to the other as $|\delta x|$ is greater than $c\,|\delta t|$. This can be checked for all the numerical examples given above. It will be shown in Chapter 5 that, according to the theory of special relativity, energy and momentum cannot be transmitted with a velocity exceeding the velocity of light. Hence, when δs^2 is positive, what happens in one of the events cannot influence what happens in the other as light cannot go from one event to the other. There can be no direct causal connection between the two events, so that it is irrelevant which event is measured to occur first, as neither event can influence the other. One cannot influence what is going on at present on a star a thousand light years away, since, if one shines a torch at the star now, the light will not reach the star until 1000 years hence. One cannot even influence what is going on on the moon at present, because a light signal would take 1·28 sec to reach the moon and a rocket would take much longer. When δs^2 is negative, that is δs is a time-like interval, then it is possible to send a light signal from one event to the other since $|\delta x| < c\,|\delta t|$, as illustrated in the numerical examples. In these cases causal connection between the two events is possible. For example, one could influence what happens on the moon in a year's time by sending a rocket to reach there in time. For events for which δs^2 is negative there is a definite time order for the events, which cannot be reversed by transforming to any other inertial frame moving with velocity $v < c$ relative to the first. These conclusions are illustrated graphically in *Figure 3.14*. It will again be assumed that the zeros of x and t are chosen such that one event takes place at the origin of the inertial frame Σ at $t = 0$. The abscissa represents the x co-ordinate of the other event and the ordinate represents the

product of the velocity of light and the time of the other event. The paths of light rays going along the x axis and which are at the origin $x = 0$, $t = 0$ are lines given by $x = \pm ct$ and are at 45 degrees to the axes as shown in *Figure 3.14*. Events in the region labelled FUTURE can be reached from the origin with velocities less than the velocity of light and can be influenced by what happens at the origin at $t = 0$. Events in the region labelled PAST could

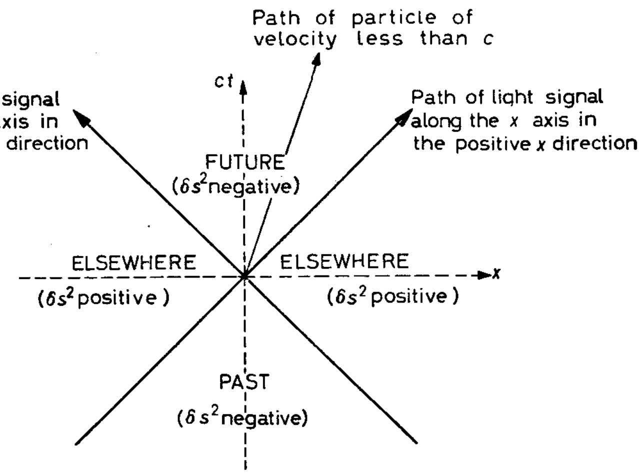

Figure 3.14. One event takes place at the origin at $x = 0$, $t = 0$. If another event takes place in the regions marked PAST *and* PRESENT, *then δs^2, the interval between the two events, is negative and causal connection between the two events is possible. If the other event takes place in the regions marked* ELSEWHERE *then δs^2 is positive and no causal connection is possible between the event and the event at the origin*

have sent signals to reach the origin at or before $t = 0$, and could influence what happens at $x = 0$, $t = 0$. Events in the regions labelled ELSEWHERE cannot send light signals to reach the origin by the time $t = 0$, neither can light signals be sent from the origin to reach these events in time to have any influence on them, so that these events can have no causal connection with what happens in the event at the origin at $x = 0$, $t = 0$.

The intervals between the event at the origin and events in either the PAST or FUTURE regions are time-like ($\delta s^2 < 0$), and the intervals between the event at the origin and all events in the ELSEWHERE regions are space-like ($\delta s^2 > 0$). Future and past take on a new meaning in the theory of special relativity depending on whether or not causal connection is or was possible. With a

little imagination on the part of the reader, *Figure 3.14* can be extended to events, for which the y and z co-ordinates of the events are not zero.

3.13. THE ROLE OF OBSERVERS IN THE THEORY OF SPECIAL RELATIVITY

In this chapter we have often discussed what an observer at rest in the inertial frame Σ or an observer at rest in the inertial frame Σ' would measure. For example in Section 3.6, we considered what an observer on rocket 1 and what an observer on rocket 2 would measure. The use of the term observers in such contexts led some people to conclude that the theory of special relativity required the intervention of observers in measurements in a way different from classical Newtonian mechanics, and for this reason the theory of relativity was assumed by many people to be more 'idealistic' and less 'materialistic' than Newtonian mechanics. In both Newtonian mechanics and in the theory of special relativity the measurements can be performed by instruments, and the information transmitted to a base where it can be fed into a calculating machine, and after constructing and setting up the apparatus the 'observer' need only read and interpret the results when necessary. The role of the observer is very similar in both theories, so that on this point one theory cannot be said to be more idealistic or materialistic than the other. It is beyond the scope of this work to discuss this aspect of the two theories any further. The observers were introduced in the previous discussion of Newtonian mechanics in Section 1.4 and in the present chapter merely to try and make the exposition easier to understand. In future we shall generally merely state that a variable is measured to have a certain value in Σ rather than say that an observer in Σ would measure it to have that particular value, unless it helps the exposition to introduce imaginary observers.

What the theory of special relativity did show was that some quantities such as length and time intervals, which had previously been assumed *a priori* to have invariant values in all inertial frames of reference, need not have the same numerical values in different inertial frames moving relative to each other, but their numerical values may depend on the arbitrary standard of rest chosen. It was illustrated in Sections 3.6 and 3.7 that these predictions are plausible when they are interpreted in terms of measurements that can actually be carried out in the laboratory. The theory of special relativity emphasized that the quantities used in a theory must be defined and measured in terms of that particular theory, and the

quantities used in a theory must be related directly or indirectly to measurements that can be performed in the laboratory.

For purposes of discussion it was found convenient to think of the co-ordinate system Σ and Σ' as consisting of a series of 'rigid' rulers parallel to the axes, with a series of synchronized clocks distributed throughout space so that the co-ordinates and time of an event could be measured when and where it happened. It must be stressed that this model was a purely symbolic way of illustrating how the co-ordinates and time of an event might be measured. One could perform the measurements in practice using the radar techniques described in Section 3.9 and Appendix 6.

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PROBLEMS

Problem 3.1—What are the two main postulates of the theory of special relativity? What extra postulates are implicit in the theory of special relativity?

Problem 3.2—Write a critical essay on what is meant by 'an inertial frame of reference' in various physical theories. Discuss the scope and limits of applicability of the concept.

Problem 3.3—A rod of proper length L_0 is at rest in an inertial frame Σ'. The rod is inclined at an angle θ' to the x' axis in Σ'. What is the length of

the rod in Σ, and what is its inclination to the x axis in Σ, if Σ' moves with uniform velocity v relative to Σ along the common x axis?

Problem 3.4—By what amount is the earth shortened along its diameter (as measured by an observer at rest relative to the sun) due to the orbital motion of the earth around the sun? (Take the velocity of the earth as 30 km/sec and the radius of the earth as 6367 km.)

Problem 3.5—A rocket is moving at such speed in the lab.-system that its measured length is half its proper length. How fast is the rocket moving relative to the lab.-system?

Problem 3.6—Show that the circle $x'^2 + y'^2 = a^2$ in Σ' is measured to be an ellipse in Σ, if Σ' moves with uniform velocity relative to Σ.

Problem 3.7—There are clocks at rest in both Σ and Σ'. The clocks at rest in Σ are synchronized with each other, and the clocks at rest in Σ' are synchronized with each other. The origins of Σ and Σ' coincide at $t = t' = 0$, and Σ' moves with uniform velocity v along the $+x$ axis relative to Σ. At a later time t, which of the clocks at rest in Σ' read the same time as all the clocks at rest in Σ? [Hint: put $t' = t$ in the time transformation to obtain

$$x = \frac{c^2}{v}\{1 - (1 - v^2/c^2)^{\frac{1}{2}}\}\, t.$$

The clocks can have any y and z co-ordinates.]

Problem 3.8—A rocket of proper length 600 m is moving directly away from the earth with uniform velocity. A light (or radar) pulse is sent out from the earth and is reflected from mirrors at the back end and the front end of the rocket. If the first light (radar) pulse is received back at base 200 sec after emission and the second pulse is received 17·4 μ-sec later, calculate (*a*) the distance of the rocket from the earth and (*b*) the velocity of the rocket relative to the earth.

Problem 3.9—How would you measure the length of a moving rocket? [Hint: An adaptation of the method of the previous example may be suitable.]

Problem 3.10—If the mean lifetime of a μ-meson when it is at rest is $2{\cdot}2 \times 10^{-6}$ sec, calculate the average distance it will travel *in vacuo* before decay, if its velocity is (*a*) 0·9 c; (*b*) 0·99 c; (*c*) 0·999 c.

Problem 3.11—One thousand μ-mesons are produced at an altitude of 40 km. How many μ-mesons should on the average reach sea level before they decay if the velocity of the μ-mesons is 0·999c and if their mean life is $2{\cdot}2 \times 10^{-6}$ sec when they are at rest? (Assume the μ-mesons travel vertically downwards without energy loss due to ionization.)

Problem 3.12—Describe the experiment performed by Durbin, Loar and Havens [*Phys. Rev.* **88** (1952) 179] on the lifetime of moving charged π-mesons. How did the measured lifetime of moving π-mesons compare with the accepted value for the mean lifetime of π-mesons when they are at rest?

Problem 3.13—An astronaut wants to go to a star 5 light years away. The rocket accelerates quickly and then moves with uniform velocity. Calculate with what velocity the rocket must move relative to the earth, if the astronaut is to reach there in one year, as measured by clocks at rest on the rocket.

Problem 3.14—Give an account of the Ives and Stilwell experiment on the second-order Doppler effect. Explain how their results can be interpreted in terms of the Lorentz transformations.

Problem 3.15—Calculate the Doppler shift in wavelength (using the relativistic formula) for light of wavelength 6000 Å when the source is approaching the observer with velocity (*a*) $0{\cdot}1c$ and (*b*) $0{\cdot}5c$.

Problem 3.16—One event occurs at the origin of the inertial frame Σ at $t = 0$. A second event occurs at the point $x = 5c, y = z = 0$ at a time $t = 4$ sec relative to Σ. Find the velocities, relative to Σ, of the inertial reference frames in which (*a*) the events are simultaneous, (*b*) event 2 precedes event 1 by 1 sec, and (*c*) event 1 precedes event 2 by 1 sec.

Problem 3.17—One event occurs at the origin of an inertial frame Σ at the time $t = 0$. Another event occurs at $x = 4c, y = z = 0$ at a time $t = 5$ sec relative to Σ. (*a*) Determine the velocity (relative to Σ) of the inertial frame Σ' in which the two events are recorded at the same point in space, and (*b*) what is the time interval between the events in Σ'.

Problem 3.18—If two events occur at the same time $t = 0$ at points $x = y = z = 0$ and $x = X, y = z = 0$ in an inertial frame Σ, show that the interval between the two events is space-like. Show that by a suitable choice of inertial frame the spatial separation between the events can be made to vary between $+X$ and ∞. Show that there is no limit on the time separation between the events, and the time order of the events may be different in different inertial frames. Discuss how the last result led to a refinement of the classical idea of causality, according to which all earlier events could influence all later events wherever they occurred.

Problem 3.19—The point O' in an inertial frame Σ has Cartesian co-ordinates $x = vt, y = z = 0$ at time t, where v is a constant. A second frame Σ' has O' as origin and Cartesian axes parallel to those of Σ. Derive the Lorentz transformations between the co-ordinates of an event in Σ and Σ'.

In Σ, events occur at the origin and at the point $(X, 0, 0)$ simultaneously at $t = 0$. The time interval between the events in Σ' is T. Show that the spatial distance between the events in Σ' is $(X^2 + c^2T^2)^{\frac{1}{2}}$, and determine the relative velocity v of the two frames in terms of X and T. [Hint: Use the fact that the interval between the events is an invariant.] (Exeter 1960.)

4

RELATIVISTIC KINEMATICS

4.1. THE VELOCITY TRANSFORMATIONS

The Lorentz transformations for the transformation of the co-ordinates and time of an event from one inertial frame to another were derived in Chapter 3 from the principle of relativity, and the principle of the constancy of the velocity of light. In this chapter the transformations for the velocity and acceleration of a particle will be calculated on the basis of the Lorentz transformations. When a particle is moving relative to an inertial frame, its velocity is defined as the distance it moves per unit time measured in that inertial frame. Let a particle be measured to be at a point x, y, z at a time t in an inertial frame Σ, and let it be measured to be at the point $x + \delta x, y + \delta y, z + \delta z$, at a time $t + \delta t$. The velocity of the particle relative to Σ is defined as a vector $\mathbf{u}$ having components

$$u_x = \frac{\delta x}{\delta t}; \quad u_y = \frac{\delta y}{\delta t}; \quad u_z = \frac{\delta z}{\delta t} \tag{4.1}$$

The magnitude of the velocity of the particle is equal to

$$u = \sqrt{u_x^2 + u_y^2 + u_z^2} \tag{4.2}$$

The two successive measurements of the positions of the particle are events, since they involve observations, such as the passing of the particle past marks on a scale, or the passage of a high energy charged particle through two scintillation counters. These events occur independently of any particular co-ordinate system. The co-ordinates and time of these events in different inertial frames are related by the Lorentz transformations. In an inertial frame Σ', moving with uniform velocity v relative to Σ along the common x axis, corresponding to x, y, z, t one has

$$x' = \gamma(x - vt); \quad y' = y; \quad z' = z; \quad t' = \gamma(t - vx/c^2) \tag{4.3}$$

and corresponding to $x + \delta x, y + \delta y, z + \delta z, t + \delta t$ one has

$$x' + \delta x' = \gamma[x + \delta x - v(t + \delta t)]; \quad y' + \delta y' = y + \delta y$$

$$z' + \delta z' = z + \delta z; \quad t' + \delta t' = \gamma\left[t + \delta t - \frac{v(x + \delta x)}{c^2}\right] \tag{4.4}$$

Subtracting (4.3) from (4.4),

$$\delta x' = \gamma[\delta x - v\,\delta t]; \quad \delta y' = \delta y; \; \delta z' = \delta z; \; \delta t' = \gamma\left[\delta t - \frac{v\,\delta x}{c^2}\right]$$

The velocity of the particle measured in Σ' has components

$$u'_x = \frac{\delta x'}{\delta t'} = \frac{(\delta x - v\,\delta t)}{\left(\delta t - \dfrac{v}{c^2}\,\delta x\right)} = \frac{(\delta x/\delta t - v)}{\left(1 - \dfrac{v}{c^2}\dfrac{\delta x}{\delta t}\right)} = \frac{(u_x - v)}{(1 - vu_x/c^2)} \tag{4.5}$$

$$u'_y = \frac{\delta y'}{\delta t'} = \frac{\delta y\sqrt{1 - v^2/c^2}}{\left(\delta t - \dfrac{v}{c^2}\,\delta x\right)} = \frac{(\delta y/\delta t)\sqrt{1 - v^2/c^2}}{\left(1 - \dfrac{v}{c^2}\dfrac{\delta x}{\delta t}\right)} = \frac{u_y\sqrt{1 - v^2/c^2}}{(1 - vu_x/c^2)} \tag{4.6}$$

$$u'_z = \frac{\delta z'}{\delta t'} = \frac{u_z\sqrt{1 - v^2/c^2}}{(1 - vu_x/c^2)} \tag{4.7}$$

These transformations relate to the instantaneous values of the velocity in Σ and Σ' at some specific identifiable point on the path of the particle. The co-ordinates and time of this point in Σ and Σ' are related by the Lorentz transformations. Notice that, even though the increments δy and $\delta y'$ are equal, u_y and u'_y are not equal, the difference between them arising from the different measures of time intervals in Σ and Σ'. Notice, if $\mathbf{u}$ and v are very much smaller than c, then

$$u'_x \to u_x - v; \quad u'_y \to u_y \quad \text{and} \quad u'_z \to u_z$$

This is in agreement with the velocity transformations of Newtonian mechanics.

Now

$$\begin{aligned} u'^2 &= u'^2_x + u'^2_y + u'^2_z \\ &= \frac{(u_x - v)^2 + (u_y^2 + u_z^2)(1 - v^2/c^2)}{(1 - vu_x/c^2)^2} \\ &= \frac{(u_x - v)^2 + (u^2 - u_x^2)(1 - v^2/c^2)}{(1 - vu_x/c^2)^2} \end{aligned} \tag{4.8}$$

The inverse relations are obtained by interchanging primed and unprimed quantities and replacing v by $-v$. Eqns (4.5), (4.6), (4.7) and (4.8) become

$$u_x = \frac{u'_x + v}{(1 + vu'_x/c^2)} \tag{4.9}$$

$$u_y = \frac{u'_y\sqrt{1 - v^2/c^2}}{(1 + vu'_x/c^2)} \tag{4.10}$$

$$u_z = \frac{u'_z\sqrt{1 - v^2/c^2}}{(1 + vu'_x/c^2)} \tag{4.11}$$

$$u^2 = \frac{(u'_x + v)^2 + (u'^2 - u'^2_x)(1 - v^2/c^2)}{(1 + vu'_x/c^2)^2} \tag{4.12}$$

The symbols $\mathbf{u}$ and $\mathbf{u}'$ will be used for velocities measured in the inertial frames Σ and Σ' respectively. The symbol v will only be used for the relative velocity of the two inertial frames Σ and Σ', and the symbol γ will always represent $1/(1 - v^2/c^2)^{\frac{1}{2}}$.

As an example of the application of the velocity transformations, consider a ball which is rolling along the corridor of a train with a velocity u_x' relative to the train (Σ'). If the train moves with a velocity v along the x axis relative to the embankment (Σ), then, according to the transformations of the theory of special relativity, the velocity of the ball relative to the embankment should be given by

$$u_x = \frac{(u_x' + v)}{(1 + vu_x'/c^2)}$$

As a typical example, let u_x' be 5 m/sec and let v be 30 m/sec, then

$$u_x = \frac{(5 + 30)}{\left(1 + \dfrac{5 \times 30}{(3 \times 10^8)^2}\right)} = 34{\cdot}9999999999999417 \text{ m/sec}$$

According to Newtonian mechanics

$$u_x = u_x' + v = 5 + 30 = 35 \text{ m/sec}$$

This example illustrates how, in normal circumstances, the deviations from Newtonian mechanics can be neglected.

Let a radioactive nucleus move with a velocity $v = 0{\cdot}2c$ along the x axis of the laboratory system Σ. Let it emit a β-particle of velocity $u_x' = 0{\cdot}95c$ relative to the inertial frame Σ' in which the radioactive nucleus is at rest. If the β-particle is emitted along the x' axis of Σ', its speed relative to the laboratory is given by

$$u_x = \frac{u_x' + v}{1 + vu_x'/c^2} = \frac{0{\cdot}95c + 0{\cdot}2c}{1 + 0{\cdot}2 \times 0{\cdot}95} = 0{\cdot}966c$$

According to the Galilean transformations the speed should have been $0{\cdot}95c + 0{\cdot}2c = 1{\cdot}15c$. Thus, in nuclear physics, the deviations from Newtonian mechanics are very important.

If one puts $u_x' = w$ and $u_y' = u_z' = 0$ in eqns (4.9), (4.10) and (4.11), then, in the inertial frame Σ, one would have

$$u_x = \frac{w + v}{1 + vw/c^2}; \quad u_y = 0, \quad u_z = 0 \tag{4.13}$$

In Section 3.4 an inertial frame Σ'' was considered, moving with uniform velocity w relative to Σ' along the x axis, whilst Σ' moved with uniform velocity v relative to Σ along the x axis. By applying

the Lorentz transformations twice, it was shown that the origin of Σ'' moves with a velocity $(w + v)/(1 + vw/c^2)$ relative to Σ. This is in agreement with eqn (4.13).

If $u'_x = c$, then in Σ:

$$u_x = \frac{c + v}{(1 + vc/c^2)} = c$$

Thus the velocity of light *in vacuo* has the same numerical value in Σ and Σ', illustrating that the velocity transformations are consistent with the principle of the constancy of the velocity of light, as, of course, they should be.

If $u = \alpha c$ and $v = \beta c$, then

$$u_x = \frac{\alpha c + \beta c}{\left(1 + \dfrac{\beta c \alpha c}{c^2}\right)} = c\left(\frac{\alpha + \beta}{1 + \alpha\beta}\right)$$

Provided α and β are both less than unity, then $(\alpha + \beta)/(1 + \alpha\beta)$ is less than unity, as direct substitution of numerical values for α and β will show, and u_x will then always be less than c. Velocities exceeding the velocity of light cannot be obtained by the composition of a number of velocities, which are themselves less than c.

4.2. THE TRANSFORMATIONS FOR $(1 - u'^2/c^2)^{\frac{1}{2}}$ AND $(1 - u^2/c^2)^{\frac{1}{2}}$

The quantities $(1 - u'^2/c^2)^{\frac{1}{2}}$ and $(1 - u^2/c^2)^{\frac{1}{2}}$ occur very frequently in relativistic formulae. The transformations for these quantities will now be derived. From eqn (4.8)

$$u'^2 = \frac{(u_x - v)^2 + (u^2 - u_x^2)(1 - v^2/c^2)}{(1 - vu_x/c^2)^2} \tag{4.8}$$

Thus,

$$1 - u'^2/c^2 = 1 - \frac{\left[\left(\dfrac{u_x}{c} - \dfrac{v}{c}\right)^2 + \left(\dfrac{u^2}{c^2} - \dfrac{u_x^2}{c^2}\right)\left(1 - \dfrac{v^2}{c^2}\right)\right]}{(1 - vu_x/c^2)^2}$$

$$= \frac{1 - \dfrac{2vu_x}{c^2} + \dfrac{v^2u_x^2}{c^4} - \dfrac{u_x^2}{c^2} + \dfrac{2vu_x}{c^2} - \dfrac{v^2}{c^2} - \dfrac{u^2}{c^2} + \dfrac{u_x^2}{c^2} + \dfrac{v^2u^2}{c^4} - \dfrac{v^2u_x^2}{c^4}}{(1 - vu_x/c^2)^2}$$

$$= \frac{1 - v^2/c^2 - u^2/c^2 + v^2u^2/c^4}{(1 - vu_x/c^2)^2} = \frac{(1 - v^2/c^2)(1 - u^2/c^2)}{(1 - vu_x/c^2)^2}$$

Taking the square root,

$$\sqrt{1 - u'^2/c^2} = \frac{\sqrt{(1 - v^2/c^2)(1 - u^2/c^2)}}{(1 - vu_x/c^2)} \tag{4.14}$$

In eqn (4.14), u' is the total velocity of a particle measured in Σ', and u is the total velocity of the particle measured in Σ. The left-hand side of eqn (4.14) involves only quantities measured in Σ', and apart from v, the right-hand side contains only quantities measured in Σ. Eqn (4.14) is the transformation for $(1 - u'^2/c^2)^{\frac{1}{2}}$.

Similarly, for the inverse relation,

$$\sqrt{1 - u^2/c^2} = \frac{\sqrt{(1 - v^2/c^2)(1 - u'^2/c^2)}}{(1 + vu'_x/c^2)} \tag{4.15}$$

Eqns (4.14) and (4.15) will be used extensively throughout the text and the reader should be thoroughly familiar with them.

4.3. THE TRANSFORMATIONS FOR THE ACCELERATION OF A PARTICLE

In the inertial frame Σ the acceleration $\mathbf{a}$ of a particle is defined as the rate of change of the velocity of the particle. It is a vector having components:

$$a_x = \frac{\mathrm{d}u_x}{\mathrm{d}t}; \quad a_y = \frac{\mathrm{d}u_y}{\mathrm{d}t}; \quad a_z = \frac{\mathrm{d}u_z}{\mathrm{d}t} \tag{4.16}$$

Similarly, the acceleration $\mathbf{a}'$ measured in Σ' is a vector having components

$$a'_x = \frac{\mathrm{d}u'_x}{\mathrm{d}t'}; \quad a'_y = \frac{\mathrm{d}u'_y}{\mathrm{d}t'}; \quad a'_z = \frac{\mathrm{d}u'_z}{\mathrm{d}t'} \tag{4.17}$$

Now

$$a'_x = \frac{\mathrm{d}u'_x}{\mathrm{d}t'} = \frac{\mathrm{d}u'_x}{\mathrm{d}t} \times \frac{\mathrm{d}t}{\mathrm{d}t'} = \frac{\mathrm{d}u'_x/\mathrm{d}t}{\mathrm{d}t'/\mathrm{d}t}$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{(u_x - v)}{(1 - vu_x/c^2)}\right\} \bigg/ \frac{\mathrm{d}}{\mathrm{d}t}\,[\gamma(t - vx/c^2)]$$

It is being assumed that Σ' moves with uniform velocity relative to Σ, so that v and γ are constants. Carrying out the differentiation and remembering that $a_x = \mathrm{d}u_x/\mathrm{d}t$ etc., one obtains

$$a'_x = \frac{1}{\gamma\left(1 - \dfrac{vu_x}{c^2}\right)}\left[\frac{(1 - vu_x/c^2)a_x - (u_x - v)(-1)va_x/c^2}{(1 - vu_x/c^2)^2}\right]$$

$$= \frac{a_x\left(1 - \dfrac{vu_x}{c^2} + \dfrac{vu_x}{c^2} - v^2/c^2\right)}{\gamma\left(1 - \dfrac{vu_x}{c^2}\right)^3}$$

that is

$$a_x' = \frac{(1 - v^2/c^2)^{\frac{3}{2}}}{\left(1 - \dfrac{vu_x}{c^2}\right)^3} a_x \tag{4.18}$$

$$a_y' = \frac{\mathrm{d}u_y'}{\mathrm{d}t'} = \frac{\mathrm{d}u_y'}{\mathrm{d}t} \Big/ \frac{\mathrm{d}t'}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{u_y}{\gamma\left(1 - \dfrac{vu_x}{c^2}\right)}\right] \Big/ \frac{\mathrm{d}}{\mathrm{d}t}\left[\gamma\left(t - \frac{vx}{c^2}\right)\right]$$

$$= \frac{1}{\gamma\left[1 - \dfrac{vu_x}{c^2}\right]} \frac{\left[\left(1 - \dfrac{vu_x}{c^2}\right)a_y + u_y \dfrac{va_x}{c^2}\right]}{\gamma\left(1 - \dfrac{vu_x}{c^2}\right)^2}$$

Hence,

$$a_y' = \frac{\left(1 - \dfrac{v^2}{c^2}\right)}{\left(1 - \dfrac{vu_x}{c^2}\right)^2}\left\{a_y + \frac{vu_y/c^2}{(1 - vu_x/c^2)} a_x\right\} \tag{4.19}$$

Similarly,

$$a_z' = \frac{\mathrm{d}u_z'}{\mathrm{d}t'} = \frac{(1 - v^2/c^2)}{\left(1 - \dfrac{vu_x}{c^2}\right)^2}\left\{a_z + \frac{vu_z/c^2}{(1 - vu_x/c^2)} a_x\right\} \tag{4.20}$$

The transformations (4.18), (4.19) and (4.20) relate the instantaneous values of the acceleration of a particle, measured in Σ and Σ' respectively, at some identifiable point on the path of the particle; the co-ordinates and time of this point in Σ and Σ' are related by the Lorentz transformations.

If $\mathbf{u}$ and v are very much smaller than c, then $\mathbf{a}' = \mathbf{a}$. This is in agreement with Newtonian mechanics, according to which theory the acceleration of a particle should be the same in all inertial frames moving with uniform velocity relative to each other [compare eqn (1.13)].

If in the inertial frame Σ, a particle moves with uniform acceleration a_x in the x direction, then u_x, the velocity of the particle in the x direction, varies with time. The expression on the right-hand side of eqn (4.18) includes a term in u_x, which varies with time, so that a_x' does not remain constant even if a_x remains constant.

In Newtonian mechanics the mass of a particle was assumed to be an invariant independent of the velocity of the particle. It was then possible to define the force acting on a particle as the product of the mass and the acceleration of the particle. It will be shown in

Chapter 5 that in relativistic mechanics the mass of a particle must depend on its velocity, and it is not convenient to define the force acting on a particle as the product of the mass of the particle and the acceleration produced by the force. The discussion of the relativistic definition of force is deferred until Chapter 5.

4.4. RELATIVITY OPTICS

4.4.1. *Fizeau's Experiment*

In 1851, Fizeau showed that the speed of light in moving water was affected by the motion of the water. A more accurate experiment was carried out by Michelson and Morley in 1886, who

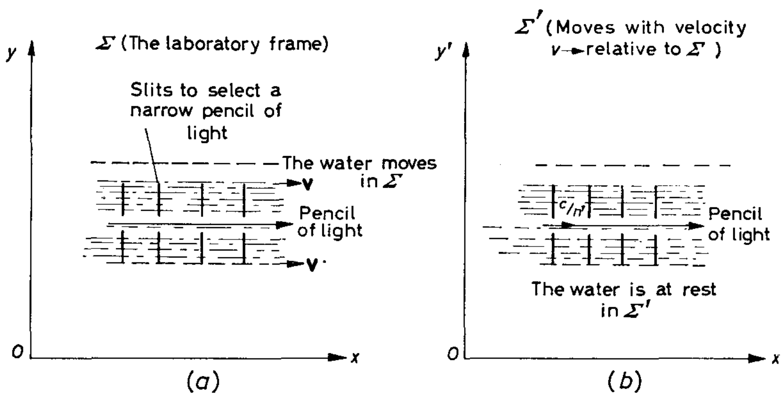

Figure 4.1. In Σ' the water is at rest, and the velocity of light is the same in all directions and is equal to c/n'. (a) In Σ the water is moving with velocity v. According to the velocity transformations of the theory of special relativity the velocity of light in the direction of motion of the water is equal to $(c/n') + v\left(1 - \frac{1}{n'^2}\right)$ in Σ

showed experimentally that, if n' is the refractive index of light in stationary water, the speed of light in water moving with velocity v was equal to $c/n' + v(1 - 1/n'^2)$ when the light travelled in a direction parallel to the direction of water flow. (See Rosser[1] Section 2.1.7.) This result will now be interpreted in terms of the velocity transformations of the theory of special relativity.

Let Σ be the laboratory system in which the water is moving with uniform velocity v in the positive x direction. Let a narrow pencil of light be selected by a series of slits such that the pencil of light travels in the positive x direction as shown in *Figure 4.1*(a). In the inertial frame Σ' moving with uniform velocity v relative to Σ the water is at rest as shown in *Figure 4.1*(b). Let the refractive

index of the light in Σ' be equal to n'. The velocity of light in Σ' is given by

$$u'_x = c/n'; \quad u'_y = 0; \quad u'_z = 0$$

Applying eqns (4.9), (4.10) and (4.11) one obtains

$$u_y = u_z = 0$$

$$u_x = \frac{(u'_x + v)}{(1 + vu'_x/c^2)} = \frac{\left(\dfrac{c}{n'} + v\right)}{\left(1 + \dfrac{vc}{n'c^2}\right)} = \frac{c}{n'}\left\{1 + \frac{n'v}{c}\right\}\left\{1 + \frac{v}{n'c}\right\}^{-1}$$

Expanding by the binomial theorem and neglecting terms of the order of v^2/c^2:

$$u_x = \frac{c}{n'}\left\{1 + \frac{n'v}{c}\right\}\left\{1 - \frac{v}{n'c}\right\}$$

$$u_x = \frac{c}{n'} + v\left\{1 - \frac{1}{n'^2}\right\} + \text{higher order terms} \tag{4.21}$$

This value for the speed of light in moving water is in agreement with the results obtained by Fizeau and by Michelson and Morley. This shows that the velocity transformations of the theory of special relativity give the correct velocity relative to the laboratory. If the Galilean velocity transformations were applied, one would have

$$u_x = u'_x + v = \frac{c}{n'} + v$$

This value is not in agreement with the experimental results. In order to account for eqn (4.21) in terms of the classical ether theories, it was necessary to postulate that the moving water dragged some of the hypothetical ether inside the water along with it.

An interesting feature of the present result is that it was not necessary to discuss the actual processes taking place in the water which give rise to the value of $(c/n') + v\left(1 - \dfrac{1}{n'^2}\right)$ for the velocity of light in moving water. It can be seen that relativistic considerations alone demanded the above result. In order to account for the result in terms of electromagnetic theory, Lorentz extended Maxwell's equations to the case of moving media and introduced hypotheses about the structure of matter. However, if the equations of the new theory obey the principle of relativity and remain invariant in mathematical form when the co-ordinates and time are changed according to the Lorentz transformations (which Lorentz's equations did to first order of v/c), then the values for velocities must

transform according to the transformations of the theory of special relativity and not Newtonian mechanics. Such a state of affairs often arises in the theory of special relativity, when the requirement that the equations of a theory be consistent with the postulates of the theory of special relativity is sufficient to yield some results, without having to discuss in detail processes actually taking place.

4.4.2. *The Aberration of the Light from Stars*

The phenomenon of aberration was discovered by the astronomer Bradley in 1728. Bradley observed a variation in the apparent position of a star at different times of the year. The phenomenon of aberration will now be interpreted in terms of the velocity transformations of the theory of special relativity.

Consider the inertial frame Σ in which the sun and the star are at rest. Choose the directions of the y and z axes such that the star lies in the xy plane. Let the earth move with velocity v relative to Σ along the positive x axis as shown in *Figure 4.2(a)*. In the inertial frame Σ', the earth is at rest as shown in *Figure 4.2(b)*; thus Σ' represents the geocentric system. The telescope selects a pencil of light which travels with the ray velocity. Let the inclination of the light relative to the x axis be equal to α and α_1' in Σ and Σ' respectively as shown in *Figures 4.2(a)* and *4.2(b)*. The telescope must be pointed at an inclination α_1' in Σ' for the light to enter the telescope along the normal to the telescope. In Σ the light velocity has components

$$u_x = -c\cos\alpha; \quad u_y = -c\sin\alpha; \quad u_z = 0$$

According to the velocity transformations

$$u_x' = \frac{u_x - v}{(1 - vu_x/c^2)} = \frac{-c\cos\alpha - v}{1 + \dfrac{vc}{c^2}\cos\alpha} \tag{4.22}$$

$$u_y' = \frac{u_y\sqrt{1 - v^2/c^2}}{\left[1 - \dfrac{vu_x}{c^2}\right]} = \frac{-c\sin\alpha\sqrt{1 - v^2/c^2}}{\left[1 + \dfrac{vc\cos\alpha}{c^2}\right]} \tag{4.23}$$

Dividing eqn (4.23) by eqn (4.22) one obtains

$$\tan\alpha_1' = \frac{u_y'}{u_x'} = \frac{c\sin\alpha\sqrt{1 - v^2/c^2}}{(1 + \dfrac{v}{c}\cos\alpha)}\;\frac{1 + (v/c)\cos\alpha}{(c\cos\alpha + v)}$$

$$\tan\alpha_1' = \frac{\sin\alpha\sqrt{1 - v^2/c^2}}{\cos\alpha + v/c} = \tan\alpha\left[\frac{\sqrt{1 - v^2/c^2}}{1 + \dfrac{v}{c}\sec\alpha}\right] \tag{4.24}$$

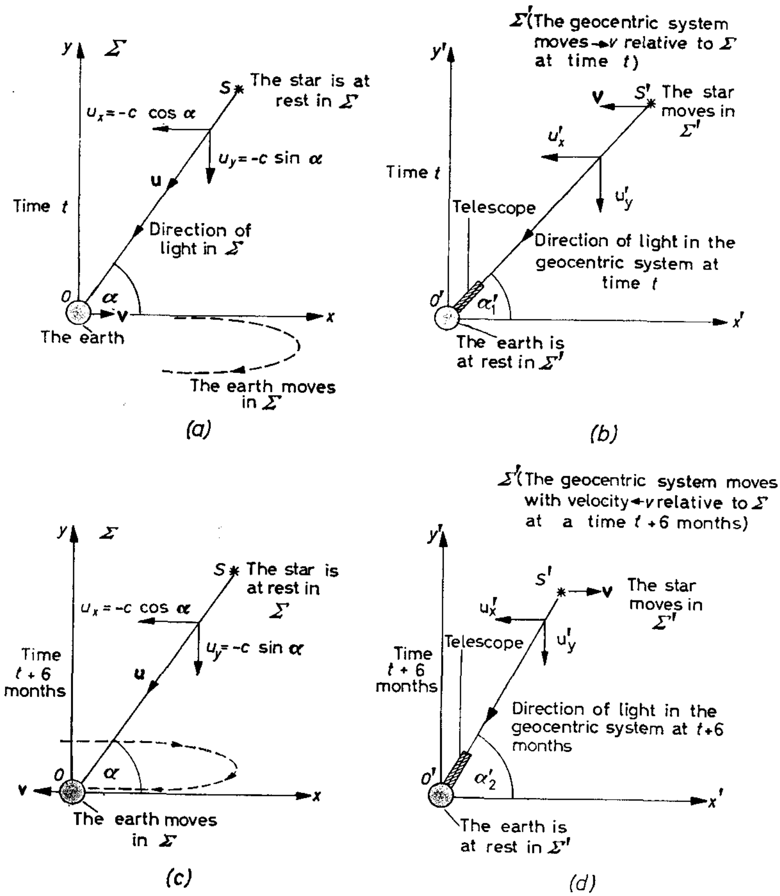

Figure 4.2. (a) The earth is moving with velocity $+v$ relative to the sun. (b) The geocentric system corresponding to (a); the telescope must be set at an inclination α_1' to the x' axis if the light from the star is to enter the telescope normally in the geocentric system. (c) Six months later, the earth is moving in the opposite direction relative to the sun. (d) The telescope must now be set at a different inclination namely α_2' in the geocentric system if the light is to enter the telescope normally in the new geocentric system

The angle α_1' is the angle the telescope must be set in the geocentric system in order that the ray of light enters the telescope along the axis of the telescope, when the earth is moving in the direction shown in *Figure 4.2(a)*. Note α_1' is less than α. Six months later the earth is moving in the opposite direction relative to the sun as shown in *Figure 4.2(c)*. This change in direction is equivalent to moving the

telescope from an inertial frame moving with velocity $+v$ relative to Σ, to one moving with velocity $-v$ relative to Σ. After six months the light from the star comes in at a different angle α_2' in the geocentric system, where α_2' is given by

$$\tan \alpha_2' = \frac{\sin \alpha \sqrt{1 - v^2/c^2}}{\cos \alpha - v/c} = \tan \alpha \left[\frac{\sqrt{1 - v^2/c^2}}{1 - \dfrac{v}{c} \sec \alpha} \right] \tag{4.25}$$

Eqn (4.25) is obtained by replacing v by $-v$ in eqn (4.24). Since α_2' is not equal to α_1', the inclination of the telescope, which is at rest on the earth, must be changed in the course of six months, if the star is to remain in the field of view. Now $v \sim 30$ km/sec $\sim 3 \times 10^4$ m/sec, whereas $c = 3 \times 10^8$ m/sec, hence terms of the order of v^2/c^2 can be neglected. Hence eqn (4.24) becomes

$$\tan \alpha_1' = \frac{\sin \alpha}{\cos \alpha + v/c} = \tan \alpha \left(1 + \frac{v}{c \cos \alpha}\right)^{-1}$$

Expanding by the binomial theorem and neglecting terms of the order v^2/c^2

$$\tan \alpha_1' = \tan \alpha - \frac{v \tan \alpha}{c \cos \alpha}$$

that is,

$$\tan \alpha_1' - \tan \alpha = -\frac{v \tan \alpha}{c \cos \alpha} \tag{4.26}$$

Let $\tan \alpha_1' = \tan (\alpha + \Delta\alpha)$ where $\Delta\alpha \ll \alpha$.

Expanding in a Taylor series

$$\tan (\alpha + \Delta\alpha) \simeq \tan \alpha + \Delta\alpha \frac{\mathrm{d}}{\mathrm{d}\alpha} (\tan \alpha) = \tan \alpha + \Delta\alpha \sec^2 \alpha$$

Hence

$$\tan (\alpha + \Delta\alpha) - \tan \alpha = \Delta\alpha \sec^2 \alpha$$

Comparing with eqn (4.26),

$$\Delta\alpha \sec^2 \alpha = -\frac{v \tan \alpha}{c \cos \alpha}$$

or

$$\Delta\alpha = -(v/c) \sin \alpha$$

This means that α_1' is less than α, so that the telescope has to be pointed at an angle less than α to the horizontal in the geocentric system, when the earth is moving in the direction shown in *Figure 4.2(a)*.

Similarly, six months later

$$\Delta\alpha = +(v/c) \sin \alpha$$

In this case the inclination of the telescope to the horizontal is greater than α, and greater than the inclination six months earlier. Due to the motion of the earth around the sun the inclination of the telescope has to be changed between the limits given by eqns (4.25) and (4.24) if the star is to remain in the centre of the field of view. If the star is overhead, the star appears to move in a circle for which $\Delta\alpha = v/c = 10^{-4}$ rad $= 20{\cdot}5$ seconds of arc. This result is in agreement with the experimental value and also with the value derived on the basis of the classical ether theories.

4.4.3. *The Transformation of Plane Waves* in Vacuo

The transformation of a plane wave *in vacuo* will now be considered using the transformations of the theory of special relativity. Without loss of generality the y and z axes can be chosen such that

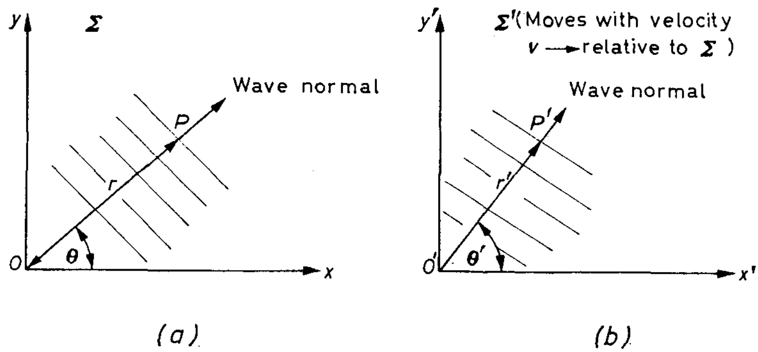

Figure 4.3. The transformation of a plane wave in vacuo. *The direction of the wave normal is different in Σ and Σ'*

the normal to the wavefront lies in the xy plane. If the normal makes an angle θ with the x axis in Σ as shown in *Figure 4.3(a)*, then the monochromatic plane wave can be represented by

$$\psi = A \cos 2\pi\nu(t - r/c) = A \cos 2\pi\nu\left(t - \frac{x \cos \theta + y \sin \theta}{c}\right)$$

where ν is the frequency of the light, c is the phase velocity of light *in vacuo*, ψ can represent either the electric intensity vector $\mathbf{E}$ or the magnetizing force $\mathbf{H}$. The phase $(t - r/c)$ can be interpreted as follows. Let a wave crest pass the origin of Σ at the time $t = 0$

and let this wave crest be given a label. Let an observer at rest at the point P in Σ start counting when this labelled wave crest reaches him at a time r/c later, and let him continue to count until the time t. Since he counts for a time $t - r/c$, the *number* of wave crests he counts is equal to $\nu(t - r/c)$.

Now, let the inertial frame Σ' move with uniform velocity v relative to Σ along the common x axis as shown in *Figure 4.3*(b), and let the origins of Σ and Σ' coincide at $t = t' = 0$. Let an observer be at rest at a point P' in Σ', and let P' coincide with P at the time t measured in Σ. Let the corresponding time measured in Σ' be t'. Let the observer at P' start counting the number of wave crests reaching him between the time r'/c when the labelled wave crest passes him and the time t'. The number of wave crests counted in Σ' will be equal to $\nu'(t' - r'/c)$, where ν' is the frequency of the light measured in Σ', and $O'P'$ is equal to r'. It is being assumed that the velocity of light *in vacuo* is the same in Σ and Σ'. Now the number of wave crests between the labelled wave crest and the wave crest passing the points P and P' at the instant they coincide is a pure number and is an invariant and independent of any particular co-ordinate system. Hence the phase must be an invariant, and for a plane wave *in vacuo* one must have

$$\nu\left(t - \frac{x\cos\theta + y\sin\theta}{c}\right) = \nu'\left(t' - \frac{x'\cos\theta' + y'\sin\theta'}{c}\right) \tag{4.27}$$

The co-ordinates and time in eqn (4.27) refer to the coincidence of P and P', and these co-ordinates and time are related by the Lorentz transformations. Substituting for x, y and t on the left-hand side of eqn (4.27) one obtains

$$\begin{aligned}\text{l.h.s.} &= \nu\left[\gamma\left(t' + \frac{vx'}{c^2}\right) - \frac{\gamma(x' + vt')\cos\theta}{c} - \frac{y'}{c}\sin\theta\right]\\ &= \nu\left[\gamma t'\left(1 - \frac{v}{c}\cos\theta\right) - \frac{\gamma x'}{c}\left(\cos\theta - \frac{v}{c}\right) - \frac{y'}{c}\sin\theta\right]\end{aligned}$$

This expression has the same mathematical form as the right-hand side of eqn (4.27), confirming that a plane wave in Σ transforms into a plane wave in Σ'. Since the two expressions must be identical, the coefficients of x', y' and t' must be the same in both expressions. Hence,

$$\gamma\nu(\cos\theta - v/c) = \nu'\cos\theta' \tag{4.28}$$

$$\nu\sin\theta = \nu'\sin\theta' \tag{4.29}$$

$$\gamma\nu[1 - (v/c)\cos\theta] = \nu' \tag{4.30}$$

Eqn (4.30) is the expression for the relativistic Doppler effect. Dividing eqn (4.29) by eqn (4.28) one obtains

$$\tan \theta' = \frac{\sqrt{1 - v^2/c^2} \sin \theta}{\cos \theta - v/c} = \frac{\sqrt{1 - v^2/c^2} \tan \theta}{1 - (v/c) \sec \theta} \tag{4.31}$$

Dividing (4.29) by (4.30):

$$\sin \theta' = \frac{\sqrt{1 - v^2/c^2} \sin \theta}{1 - (v/c) \cos \theta} \tag{4.32}$$

The inverse relations can be obtained by interchanging primed and unprimed quantities and by replacing v by $-v$. Eqn (4.31) gives the inclination θ' of the wave normal relative to the x' axis in Σ' at the point P' at the time t'. It can be seen that θ' and θ are not equal, so that the inclination of the wave normal is different in Σ and Σ'.

Equation (4.31) can be applied to calculate the aberration of a ray of light from a distant star. In order to apply eqn (4.31) to the case shown in *Figures 4.2(a)* and *4.2(b)*, the direction of the ray of light shown in *Figure 4.3* must be reversed so that the light comes towards the origin, that is θ in eqn (4.31) must be replaced by $+(\pi + \alpha)$ and θ' by $+(\pi + \alpha_1')$. The equation

$$\tan \theta' = \frac{\sqrt{1 - v^2/c^2} \sin \theta}{\cos \theta - v/c} \tag{4.31}$$

becomes

$$\tan (\pi + \alpha_1') = \frac{\sqrt{1 - v^2/c^2} \sin (\pi + \alpha)}{\cos (\pi + \alpha) - v/c}$$

or

$$\tan \alpha_1' = \frac{\sqrt{1 - v^2/c^2} \sin \alpha}{\cos \alpha + v/c} \tag{4.33}$$

This is in agreement with eqn (4.24). Similarly if v changes sign, eqn (4.31) becomes the same as eqn (4.25).

4.4.4. *The Doppler Effect*

In Section 3.10.2 the Doppler effect was calculated for the simple case when the relative velocity between the source and the observer was along the line joining them. The general case will now be considered. Consider a point source of light which is at rest at the origin of Σ'. In Σ the source is moving with uniform velocity v along the x axis. Consider the light emitted at the instant when the origins of Σ and Σ' coincide at $t = t' = 0$. Let this light reach the observer, who is at rest at the point P in Σ, at a time t, and let this

event be at a point P' in Σ' at a time t', as shown in *Figures 4.4(a)* and *4.4(b)*. The spherical wave in Σ can be represented by

$$\psi = \frac{A}{r} \cos 2\pi\nu(t - r/c)$$

Let the spherical wave in Σ' be represented by

$$\psi' = \frac{A'}{r'} \cos 2\pi\nu'(t' - r'/c)$$

The frequency of the light measured in the inertial frame Σ', in which the source is at rest is denoted by ν' and the frequency of the

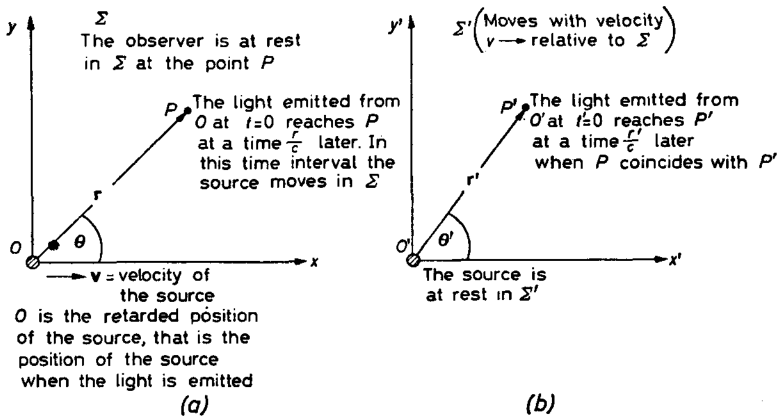

Figure 4.4. The Doppler effect. The light emitted by the moving source when it is at the origin of Σ at $t = 0$ reaches the observer, who is at rest at the point P in Σ, at a time $t = r/c$. In Σ' the source is at rest

light measured in the inertial frame Σ in which the source is moving when the light is emitted is denoted by ν. The phase is again an invariant. Proceeding precisely as in Section 4.4.3 one obtains

$$\tan \theta' = \frac{\sqrt{1 - v^2/c^2} \sin \theta}{(\cos \theta - v/c)} \tag{4.34}$$

$$\sin \theta' = \frac{\sqrt{1 - v^2/c^2} \sin \theta}{1 - (v/c) \cos \theta} \tag{4.35}$$

$$\nu' = \gamma\nu(1 - (v/c) \cos \theta)$$

or

$$\nu = \frac{\nu'\sqrt{1 - v^2/c^2}}{1 - (v/c) \cos \theta} \tag{4.36}$$

Equation (4.36) gives the relativistic Doppler effect in the general case when the velocity of the source is not necessarily along the line joining the source and the observer. Now $v \cos \theta$ is the component of the velocity of the source in the direction of the observer at the instant when the light was emitted by the source, that is, when it was at the origin. By the time the light reaches the observer the source will have moved, and in this time interval the source may undergo changes in its motion. It is generally assumed that once the light quanta leave the source, their frequencies are unaffected by any subsequent changes in the motion of the source. The position of the source at the instant when the light is emitted is called the retarded position of the source.

The transformations for the wavelength can be obtained from eqn (4.36) using $\lambda'\nu' = \lambda\nu = c$.

When the source is approaching the observer in *Figure 4.4(a)*, $\theta = 0$, $\cos \theta = +1$, and eqn (4.36) reduces to

$$\nu = \frac{\nu'\sqrt{1 - v^2/c^2}}{(1 - v/c)} = \nu'\sqrt{\frac{c + v}{c - v}} \tag{4.37}$$

The frequency of the light goes up when the source is approaching the observer. Eqn (4.37) is in agreement with eqn (3.32), which was derived in Section 3.10 directly from the Lorentz transformations.

Similarly, if the source is moving directly away from the observer when the light is emitted, then $\theta = \pi$, $\cos \pi = -1$, and eqn (4.36) becomes

$$\nu = \frac{\nu'\sqrt{1 - v^2/c^2}}{1 + v/c} = \nu'\sqrt{\frac{c - v}{c + v}} \tag{4.38}$$

In this case the frequency goes down. Eqn (4.38) is in agreement with eqn (3.28). It was described in Section 3.10 how Ives and Stilwell[2] confirmed the relativistic formulae for the Doppler effect when $\theta = 0$ and $\theta = \pi$.

If, in the inertial frame Σ, the light reaches the observer in the direction which according to the observer at P in *Figure 4.4(a)* is at 90 degrees to the direction of relative motion in the retarded position of the source, then putting $\cos \theta = 0$ in eqn (4.36), one obtains

$$\nu' = \gamma\nu$$

or

$$\nu = \nu'\sqrt{1 - v^2/c^2}$$

According to the theory of special relativity, if a beam of atoms which is emitting light is observed in a direction which according to the observer is at right angles to the direction of relative motion, then the frequency of the light should differ from the frequency the

light would have if the source were at rest relative to the observer. This is the transverse Doppler effect. According to the classical ether theories there should be no change in frequency in this case. On relativity theory the difference arises from the different measures of time in Σ and Σ'. The experimental confirmation of the transverse Doppler effect would be further support in favour of the theory of special relativity.

4.4.5. *The Visual Appearance of Rapidly Moving Objects*

Until McCrea[3], Terrell[4], Penrose[5], and Weisskopf[6] drew attention to it, it was a popular misconception that, if one looked at an object which is moving with a speed comparable to the speed of light, then one would *see* the object contracted in the direction of motion. It must be remembered that what one sees (or photographs) at a particular instant depends on the light actually reaching the eye (or the camera) at that particular instant. These light quanta will have left different parts of the object at different times. Those quanta which come from the parts furthest away from the observer leave before the quanta coming from the parts of the object nearest to the observer. Consider a cube of side l, which is moving with a uniform velocity v, which is comparable to, but less than, the velocity of light, relative to an inertial frame Σ as shown in *Figure 4.5(a)*.

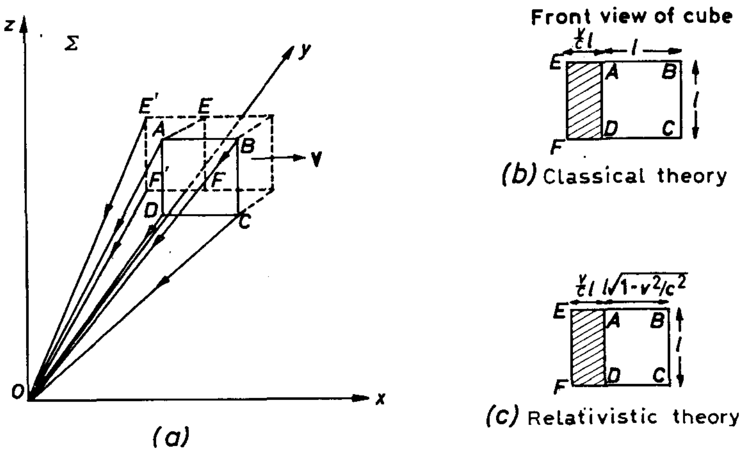

Figure 4.5. (a) The cube is moving with uniform velocity v relative to Σ. The light quanta emitted from the corners A, B, C and D, when the x co-ordinate of the cube is zero, reach O simultaneously. Light quanta emitted from the corners E and F when they are at the positions E' and F' also reach O at the same time as the quanta from A, B, C and D. The views of the cube, when it is far away from the observer, according to the classical theories and the theory of special relativity are shown in (b) and (c) respectively

Let the cube be viewed in a direction perpendicular to the direction of motion and from a large distance, so that the angle subtended by the cube at the position of the observer situated at O, the origin of Σ, is very small as shown in *Figure 4.5(a)*. We shall start by applying non-relativistic considerations. If the cube were stationary one would only see the face $ABCD$. When the cube is moving the light quanta from the four corners A, B, C and D which reach the eye at the same time to form a picture must have left the four corners at the same time. If the corners are equidistant from the point of observation when the light is emitted, as shown in *Figure 4.5(a)*, then the points A, B, C and D appear to be at the corners of a square. When the cube is moving relative to the observer, light quanta from the corners E and F can also reach the eye as shown in *Figure 4.5(a)*. The quanta from these corners leave the cube at an earlier time, when the corners E and F are at the positions E' and F' as shown in *Figure 4.5(a)*. The side $ADFE$ of the moving cube is therefore visible and the side $ADFE$ appears as a rectangle. If the eye is far away from the cube, to a first approximation, the light from E goes an extra distance l in the time that E' goes to E. Hence EE' is equal to $(v/c)l$. The non-relativistic picture of the cube is illustrated in *Figure 4.5(b)*. This view of the cube is similar to the picture of a stationary cube which has been turned through an angle, such that the face $ADFE$ is visible as shown in *Figure 4.6(a)*. Let the cube be turned through an angle α such that the projection of the side AE

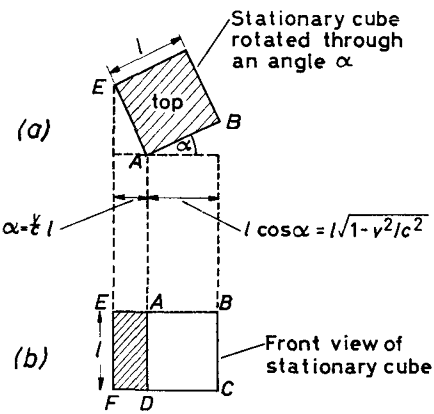

Figure 4.6. The visual appearance of a stationary cube rotated through an angle $\alpha = \sin^{-1} v/c$; the top view is shown in (a) and the front view in (b). The latter is the same as the visual appearance of a distant moving cube, predicted by the theory of special relativity and illustrated in Figure 4.5(c)

is equal to $(v/c)l$ as shown in *Figures 4.6(a)* and *(b)*. The angle α is given by

$$\sin \alpha = v/c$$

The projection of the side AB is then equal to

$$l \cos \alpha = l\sqrt{1 - v^2/c^2}$$

Thus, according to non-relativistic theory, the visual appearance of the moving cube is not quite the same as the visual appearance of the rotated stationary cube since, when the projection of the side AE is equal to $(v/c)l$, the projection of the side AB of the stationary cube is equal to $l(1 - v^2/c^2)^{\frac{1}{2}}$, whereas the side AB of the moving cube appears to have a length l. Thus according to non-relativistic theory the moving cube should have the same visual appearance as a stationary rotated cube whose length AB is increased by a factor $1/(1 - v^2/c^2)^{\frac{1}{2}}$.

According to the theory of special relativity, the lengths AE, AD DF and EF, which are all perpendicular to the direction of motion are unchanged, so that the appearance of $ADFE$ would be the same as for the non-relativistic case. On the other hand, the length AB, which is parallel to the direction of motion, should be reduced to $l(1 - v^2/c^2)^{\frac{1}{2}}$. This length contraction exactly compensates the apparent increase in the length of AB in the classical case as shown in *Figure 4.5(c)*. Thus, according to the theory of special relativity, the visual appearance of the moving cube is the same as that of a stationary cube rotated through an angle $\alpha = \sin^{-1}(v/c)$ [compare *Figures 4.5(c)* and *4.6(b)*].

It was shown by Terrell[4] that, in the general case, when the moving object is viewed at any angle then, according to the theory of special relativity, provided the angle subtended by the moving object at the point of observation is very small, the visual appearance of a moving object is undistorted in shape but rotated by the angle of aberration, where the angle of aberration can be calculated from eqn (4.35). If the object subtends a finite angle at the point of observation, then the different parts of the object appear rotated by different amounts and this leads to a distortion of shape. It was shown by Penrose[5] that the surface of a moving sphere would appear distorted if the sphere subtended a finite angle, but the outline of the moving sphere would still be circular.

One must be careful not to confuse measuring with seeing. One would measure the length of the moving cube by observing the positions of both ends of the cube on a ruler at the same time. According to the theory of special relativity one should then measure

the length to be contracted. If one *saw* an undistorted but rotated picture of a moving cube, as predicted by the theory of special relativity, then, if one calculated the dimensions of the cube allowing for the finite time of flight of the light quanta from the various parts of the cube, one would deduce that the length contraction had taken place.

The light from a moving object undergoes both aberration and the Doppler shift in frequency. It will be assumed that the moving object emits monochromatic light in the reference frame Σ' in which it is at rest. It is reasonable to assume that this light is emitted isotropically in this reference frame. Relative to Σ the object is moving with velocity v. The angular distribution of the light in Σ can be calculated from eqn (4.35). Since θ is always less than θ' except when $\theta' = \pi$, the effect of aberration is to concentrate more and more of the light in the direction of motion of the object. Thus the intensity of light emitted is greater in the forward than in the backward direction, and at extremely high velocities the light would be emitted mainly in the forward direction. When the source of light is approaching the observer, the frequency of the light is increased and for very high velocities would be shifted into the ultra-violet. After the source passes the observer the frequency drops and is shifted towards the red end of the spectrum, since the source is then moving away from the observer. Thus, when a monochromatic source of light is moving relative to the laboratory, the intensity and frequency of the light emitted in different directions varies. Thus, in addition to its apparent rotation in space, the colour and apparent brightness of a moving object are different when the object moves relative to the observer.

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PROBLEMS

Problem 4.1—A train is passing through a station at a speed of 30 m/sec. A marble is rolled along the floor of one of the compartments with a

velocity of 15 m/sec relative to the train. Calculate the speed of the marble relative to an observer standing on the platform, (*a*) if the marble rolls in the direction of motion of the train, (*b*) if the marble rolls perpendicular to the direction of motion of the train as measured by a passenger on the train.

Problem 4.2—A radioactive nucleus is moving with a velocity $c/10$ relative to the laboratory when it emits a β-particle with a velocity $0{\cdot}8c$ relative to the co-ordinate system in which the decaying nucleus is at rest. What is the velocity and direction of the β-particle relative to the laboratory, if, relative to the radioactive nucleus, it is emitted (*a*) in the direction of motion of the nucleus relative to the laboratory, and (*b*) perpendicular to the direction of motion?

If, in the laboratory system the β-particle moves in a direction perpendicular to the direction of motion of the radioactive nucleus (*c*) what is its velocity in the lab.-system and (*d*) in what direction is it emitted relative to the direction of motion of the decaying nucleus relative to the laboratory, in the co-ordinate system in which the decaying nucleus is at rest?

Problem 4.3—A radioactive nucleus which is at rest in Σ', but is moving with velocity $c/4$ along the x axis of Σ, emits a β-particle of velocity $0{\cdot}8c$ (relative to Σ') at an angle of 45 degrees to the x' axis of Σ'. What is the velocity of the β-particle relative to an observer going at a velocity of $c/4$ along the negative x axis of Σ?

Problem 4.4—An observer moving along the x axis of Σ with velocity v observes a body of proper volume V_0 moving with velocity u along the x axis of Σ. Show that the observer measures the volume to be equal to

$$V_0\sqrt{\frac{(c^2 - v^2)(c^2 - u^2)}{(c^2 - uv)^2}}$$

[Hint: The volume is $V_0\sqrt{1 - u'^2/c^2}$. Use eqn (4.14) to transform $\sqrt{1 - u'^2/c^2}$.]

Problem 4.5—A radioactive nucleus is moving with velocity $\mathbf{v}$ relative to the laboratory (Σ) when it emits a β-particle with velocity $\mathbf{u}'$ relative to the co-ordinate system Σ' in which the radioactive nucleus is at rest. The angle measured between $\mathbf{u}'$ and the x' axis of Σ' is α' (measured in Σ'). Calculate $\mathbf{u}$, the velocity of the β-particle relative to the laboratory, and calculate α, the angle between $\mathbf{u}$ and $\mathbf{v}$ in the lab.-system.

Show that if the radioactive nucleus is moving with velocity $\mathbf{u}'$ relative to the laboratory when it emits a β-particle with velocity $\mathbf{v}$, then the speed of the β-particle relative to the laboratory is the same as previously but α is not.

Problem 4.6—A physicist tells his girl friend, who weighs 20 stone (or 280 lb.) that the best way for her to slim is to move so fast relative to the laboratory that she is Lorentz contracted. (*a*) What speed would she have to move to 'reduce' her measured dimensions in the direction of motion to half her laboratory size? (*b*) What would be her mass relative to the laboratory? (Refer to Section 5.2.) (*c*) What would she really look like relative to the laboratory?

Problem 4.7—An observer at rest at the origin of an inertial frame Σ sees two particles travelling with equal speeds $0{\cdot}8c$. One goes along the positive x axis, the other along the negative x axis. (a) What is the relative velocity of the particles in Σ? How can this exceed the velocity of light? (b) With what speed are the particles transferring energy and momentum in Σ? (c) What is the velocity of one particle relative to the other?

Problem 4.8—In a cathode ray tube the speed at which the electron beam sweeps across the fluorescent screen can exceed the velocity of light in empty space. How can you reconcile this with the limiting character of the velocity of light in empty space? [Hint: Read Section 5.3, and compare with *Figure 5.2*. What is the speed of individual electrons?]

Problem 4.9—The Cartesian axes of the space-co-ordinates in the Σ'-system are parallel to those of the Σ system and the Σ' origin is at $(vt, 0, 0)$ at a time t in the Σ-system. A particle describes the parabolic trajectory $x = ut, y = \frac{1}{2}at^2$ in the Σ-system. Find its trajectory in the Σ'-system and show that its acceleration in Σ' is along $O'y'$ and of magnitude

$$a(1 - v^2/c^2)/(1 - uv/c^2)^2$$

[Hint: Use the Lorentz transformations to transform x, y and t and show

$$x' = \frac{(u - v)}{(1 - vu/c^2)}\,t'; \; y' = \tfrac{1}{2}\,\frac{a(1 - v^2/c^2)}{(1 - vu/c^2)^2}\,t'^2$$

Check on the acceleration in Σ', using the transformations for acceleration.]

Problem 4.10—A particle is at rest at the origin of an inertial frame Σ at a time $t = 0$. It moves such that it has a constant acceleration g relative to the inertial frame in which it is instantaneously at rest. Find the velocity and position of the particle relative to Σ after a time t measured in Σ.

Discussion—From eqn (4.18)

$$g = a'_x = \frac{(1 - u^2/c^2)^{\frac{3}{2}}a_x}{(1 - u^2/c^2)^3} = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{u}{\sqrt{1 - u^2/c^2}}\right]$$

Then proceed as in Section 5.4.1, except that now $g = qE/m_0$. Show that corresponding to eqn (5.36), one now has

$$u = gt\bigg/\left\{1 + \frac{g^2t^2}{c^2}\right\}^{\frac{1}{2}} \quad \text{or} \quad t = \frac{u}{g(1 - u^2/c^2)^{\frac{1}{2}}}$$

Corresponding to eqn (5.37),

$$x = \frac{c^2}{g}\left[\left\{1 + \frac{g^2t^2}{c^2}\right\}^{\frac{1}{2}} - 1\right]$$

Show that eqn (5.38) can be written as $(x + c^2/g)^2 - c^2t^2 = c^4/g^2$, or $gx^2 + 2c^2x - gc^2t^2 = 0$.

This is the mathematical equation for a hyperbola. This type of motion is consequently known as hyperbolic motion.

The motion of a charge q starting from rest in an electric field is an example of the above kind of motion. If the electric field is E_x, transforming to the reference frame Σ' in which the charge is instantaneously at

rest, one has $E'_x = E_x$ (cf. Section 7.3.3) and the force acting on it is qE'_x. Since the particle is instantaneously at rest in Σ', one can write

$$a'_x = \frac{qE'_x}{m_0} = \frac{qE_x}{m_0}$$

so that its acceleration in the inertial frame in which it is instantaneously at rest is a constant.]

Problem 4.11—Prove that light signals sent out from the origin of Σ after a time $t = c/g$ in Σ will never catch the particle discussed in the previous problem. [Hint: Show that the line $x = c(t - c/g)$ is an asymptote to the hyperbola $(x + c^2/g)^2 - c^2t^2 = c^4/g^2$.]

Problem 4.12—Consider three inertial frames Σ, Σ' and Σ''. Let Σ' move with uniform velocity v along the x axis of Σ, and let Σ'' move with uniform velocity v' along the y' axis of Σ'. Let the origins of Σ, Σ' and Σ'' coincide at $t = t' = t'' = 0$. Show that the line passing through the origins of Σ and Σ'' makes angles θ and θ'' with the x and x'' axes of Σ and Σ'', where

$$\tan\,\theta = \frac{v'(1 - v^2/c^2)^{\frac{1}{2}}}{v}$$

and

$$\tan\,\theta'' = \frac{v'}{v\sqrt{1 - v'^2/c^2}}\,.$$

If v and v' are both $\ll c$, and $v' \ll v$, show that $\theta'' - \theta \simeq vv'/2c^2$. [Hint: Apply the velocity transformations.]

Comment—This is an example of the Thomas precession. Consider an electron moving in a Bohr orbit around a nucleus. According to the theory of the Thomas precession, the axes in which the electron is instantaneously at rest appear to precess relative to the nucleus with angular frequency $\boldsymbol{\omega}$ equal to $-\mathbf{v} \times \mathbf{a}/2c^2$ where $\mathbf{v}$ is the velocity and $\mathbf{a}$ the acceleration of the electron relative to the nucleus. Derive the above equation and discuss the role it plays in the interpretation of optical spectra. Reference: Eisberg R. M. *Fundamentals of Modern Physics* p. 340.

Problem 4.13—Show that in a dispersive medium, Fresnel's dragging coefficient is equal to

$$\left\{1 - \frac{1}{n^2} - \frac{\lambda}{n}\frac{\mathrm{d}n}{\mathrm{d}\lambda}\right\}$$

where n is the refractive index appropriate to λ, the wavelength of light in the laboratory system.

For the yellow sodium D lines, the refractive index of carbon disulphide is 1·6295 and $\mathrm{d}n/\mathrm{d}\lambda = -1{\cdot}82 \times 10^5$/m. Calculate a precise value for Fresnel's dragging coefficient. You may take $\lambda_D = 5{\cdot}89 \times 10^{-7}$ m. (Rosser[1], Section 4.4.3.)

Problem 4.14—What is the Doppler shift in wavelength of the H_α (6563 Å) line from a star which is moving away from the earth with a velocity of 300 km/sec.

Problem 4.15—A light source is moving past an observer stationary in the laboratory system with a velocity of $0{\cdot}6c$. The distance of closest approach is 10 m from the observer. Find the distance (measured in the laboratory system) of the source from the observer when the source emits light which shows no Doppler shift when it reaches the observer, that is, when the frequency of the light reaching the observer, measured in the laboratory system, is equal to the frequency of the light measured in the rest frame of the source.

Problem 4.16—Assume that light is emitted isotropically in the co-ordinate system in which a sodium lamp is at rest. If the lamp moves with velocity $c/4$ relative to the laboratory, calculate the half angle of the cone containing half the emitted photons. [Hint: Consider light emitted at 90 degrees relative to the direction of motion of the lamp in the reference system in which the lamp is at rest. Calculate its direction in the lab.-system.]

Problem 4.17—A pencil of light rays is emitted into a solid angle $\mathrm{d}\Omega'$ inclined at an angle θ' to the x' axis in Σ'. Show that in Σ, which moves in the negative x' direction with uniform velocity v relative to Σ', the pencil of light rays is emitted into a solid angle $\mathrm{d}\Omega$ given by

$$\mathrm{d}\Omega = \frac{(1 - \beta^2)}{(1 + \beta \cos \theta')^2}\,\mathrm{d}\Omega', \text{ where } \beta = v/c$$

Show that the solid angle $\mathrm{d}\Omega$ is inclined at an angle θ to the x axis in Σ, where θ is given by

$$\cos \theta = \frac{\cos \theta' + \beta}{1 + \beta \cos \theta'}$$

[*Comment*—Cos θ can be obtained by dividing eqn (4.35) by eqn (4.34) and taking the inverse. We have

$$\frac{\mathrm{d}\Omega}{\mathrm{d}\Omega'} = \frac{\mathrm{d}(\cos \theta)}{\mathrm{d}(\cos \theta')}$$

If a source at rest in Σ' emits light quanta isotropically in Σ', that is the number of quanta per unit solid angle in Σ' is independent of θ' in Σ', the number/unit solid angle in Σ is proportional to $\left(\frac{\mathrm{d}\Omega'}{\mathrm{d}\Omega}\right)$. There are more quanta in the forward direction in Σ and less in the direction opposite to the direction of motion of the source. There is also a Doppler shift in frequency.]

Problem 4.18—When excited ^{57}Fe nuclei decay, they sometimes emit photons of energy 14·4 keV corresponding to a frequency of $3{\cdot}46 \times 10^{18}$ c/sec. If the ^{57}Fe nucleus moves with velocity $c/5$ when it decays via the 14·4 keV level: (*a*) calculate the frequencies of radiation emitted at angles of 0, 60, 90, 120 and 180 degrees (in the laboratory system) relative to the direction of motion of the nucleus; (*b*) by thinking of the radiation in terms of photons calculate the relative numbers of quanta per steradian at each angle in the lab.-system. [Hint: Use eqn (4.36) for part (*a*). For

part (*b*) assume the radiation is isotropic in the rest system of the decaying nucleus and use the inverse of the formula derived in *Problem 4.17*.]

Problem 4.19—A plane mirror is moving in the direction of its normal with uniform velocity v in the positive x direction in the inertial frame Σ. Let a ray of light of wavelength λ_1 strike the moving mirror at an angle of incidence α_1, and let the ray be reflected at an angle α_2 to the normal with wavelength λ_2. Show that

$$\frac{\sin \alpha_1}{\cos \alpha_1 + v/c} = \frac{\sin \alpha_2}{\cos \alpha_2 - v/c}; \quad \frac{\lambda_1}{\lambda_2} = \frac{c + v \cos \alpha_1}{c - v \cos \alpha_2}$$

[Hint: The angle of reflection is equal to the angle of incidence in the reference frame in which the mirror is at rest. Rosser[1], Section 4.4.8.]

Problem 4.20—Light of wavelength 6,000 Å is incident normally on a mirror which is receding with a velocity 3×10^7 m/sec in a direction away from the incident light. (*a*) Calculate the change in wavelength on reflection.

If the light is incident at an angle of 45 degrees to the normal of the mirror, calculate: (*b*) the change in wavelength; (*c*) the angle of reflection.

Problem 4.21—A 100 cm long 'rigid' rod moves longitudinally over a flat table. In its path is a hole 100 cm wide. Suppose the Lorentz factor of the rod relative to the table is 10. To an observer moving with the rod, the hole appears only 10 cm wide and the rod, being 'rigid', should slide over the hole. To an observer at rest at the table, however, the rod appears only 10 cm long, and in passing over the 100 cm hole it is bound to fall somewhat under gravity, and will consequently be stopped. Which view is correct? (This example is taken from Rindler W., *Special Relativity*. For a discussion of this type of problem refer to Rindler[7] and Shaw[8].)

5

RELATIVISTIC MECHANICS

5.1. INTRODUCTION

The laws of Newtonian mechanics do not obey the principle of relativity when the co-ordinates and time are changed according to the Lorentz transformations, and must be modified if mechanics is to be incorporated into the theory of special relativity. An approach due to Lewis and Tolman[1] is followed, and an imaginary experiment (a *Gedanken Experimente*) is considered in which it is assumed that two spheres collide. It is shown how Newtonian mechanics must be modified in this special case, in order to make the theory of mechanics consistent with the transformations of the theory of special relativity. The results will then be extended to more general cases. Instead of starting directly from Newton's laws, we shall start from the law of conservation of linear momentum. Alternatively one can now approach relativistic mechanics from experiments on high-speed particles, without introducing the Lorentz transformations. An outline of this approach is given in Appendix 4.

5.2. THE MASS OF A MOVING PARTICLE

It will be postulated that the law of conservation of linear momentum holds in relativistic mechanics. It will also be assumed that the mass of a body is not absolute, but may depend on the speed of the body.

Consider the collision of two small, identical, perfectly elastic spheres. It will be assumed that the spheres are small enough for the collision to be considered as taking place at approximately one point of space at one instant of time. Let sphere 1 have a velocity $\mathbf{u}'$, before the collision having components u_x', u_y', $u_z' = 0$ relative to the inertial frame Σ' shown in *Figure 5.1(b)*. Let sphere 2 have the same speed u' relative to Σ', before the collision, but let it move in the opposite direction to sphere 1, such that the components of the velocity of sphere 2 relative to Σ' are $-u_x'$, $-u_y'$, $u_z' = 0$ as shown in *Figure 5.1(b)*. Let the spheres be thrown in such a way that, when they collide, the line joining their centres is perpendicular to the x' axis of Σ'. If the collision is perfectly elastic, the total

kinetic energy should be conserved in the collision. From the symmetry of the collision, it follows that the total speeds and the x' components of the velocities of the spheres should be unchanged by the collision, but the y' components of the velocities of the spheres relative to Σ' should be reversed. Linear momentum is conserved relative to Σ' in such a collision.

The collision of the spheres will now be considered relative to the inertial frame Σ, which is moving with velocity $-v$ relative to Σ'

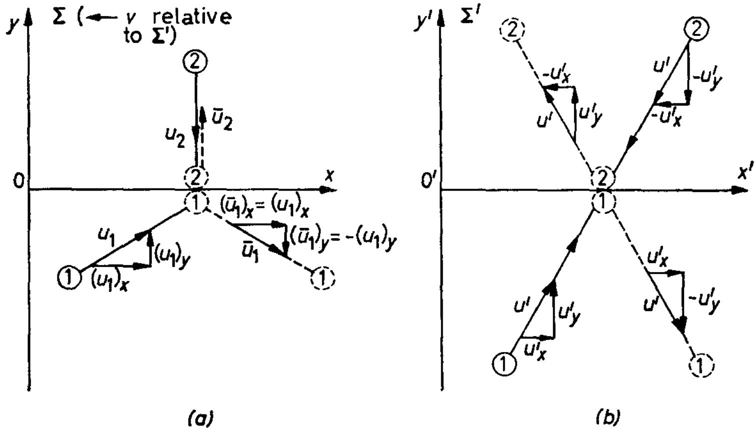


Figure 5.1. The collision of two small perfectly elastic spheres is observed from two inertial reference frames Σ and Σ'. In Σ', the spheres approach each other with equal speeds before collision. It is assumed that linear momentum is conserved relative to both Σ and Σ'

along the x' axis of Σ'. If v is numerically equal to u'_x, then, before the collision, sphere 2 moves parallel to the negative y axis of Σ as shown in *Figure 5.1(a)*. If the new laws of mechanics are to obey the principle of relativity, when the co-ordinates and time are changed according to the Lorentz transformations, then all velocities must be transformed from Σ' to Σ using the relativistic velocity transformations, namely eqns (4.9), (4.10) and (4.11).

SPHERE 1

Before collision: The x' and y' components of the velocity of sphere 1 in Σ' are $+u'_x$ and $+u'_y$ respectively, as shown in *Figure 5.1(b)*. Using eqns (4.9) and (4.10), we have for the velocity of sphere 1 in Σ before collision

$$(u_1)_x = \frac{(u'_x + v)}{(1 + vu'_x/c^2)}\,; \quad (u_1)_y = \frac{u'_y\sqrt{1 - v^2/c^2}}{(1 + vu'_x/c^2)} \tag{5.1}$$

After collision: The x' and y' components of the velocity of sphere 1 in Σ' are $+u'_x$ and $-u'_y$ respectively, as shown in *Figure 5.1*(*b*). Using eqns (4.9) and (4.10), we have for the velocity of sphere 1 in Σ after collision

$$(\bar{u}_1)_x = \frac{(u'_x + v)}{(1 + vu'_x/c^2)}\,; \quad (\bar{u}_1)_y = \frac{-u'_y\sqrt{1 - v^2/c^2}}{(1 + vu'_x/c^2)} = -(u_1)_y \quad (5.2)$$

It can be seen that the x component of the velocity of sphere 1, relative to Σ, is unchanged by the collision, whereas the y component of its velocity is reversed as shown in *Figure 5.1*(*a*). The total speed of sphere 1, relative to Σ, is unchanged by the collision.

SPHERE 2

Before collision: The x' and y' components of the velocity of sphere 2 in Σ' are $-u'_x$ and $-u'_y$ respectively as shown in *Figure 5.1*(*b*). Using eqns (4.9) and (4.10), and remembering that $u'_x = v$, we have for the velocity of sphere 2 in Σ before collision,

$$(u_2)_x = \frac{(-u'_x + v)}{(1 - vu'_x/c^2)} = 0; \quad (u_2)_y = \frac{-u'_y\sqrt{1 - v^2/c^2}}{(1 - vu'_x/c^2)} \quad (5.3)$$

After collision: The x' and y' components of the velocity of sphere 2 in Σ' are $-u'_x$ and $+u'_y$ respectively, as shown in *Figure 5.1*(*b*). Using eqns (4.9) and (4.10), since $u'_x = v$, we have for the velocity of sphere 2 in Σ after collision

$$(\bar{u}_2)_x = 0; \quad (\bar{u}_2)_y = \frac{u'_y\sqrt{1 - v^2/c^2}}{(1 - vu'_x/c^2)} = -(u_2)_y \quad (5.4)$$

Relative to Σ, the x component of the velocity of sphere 2 is zero before and after the collision. The y component of its velocity is reversed by the collision as shown in *Figure 5.1*(*a*). The total speed of sphere 2 relative to Σ is unchanged by the collision.

Let m_1 and m_2 be the masses of the spheres, before the collision, relative to Σ. If the mass of a moving body depends only on its speed, since the total speeds of the spheres are unaffected by the collision, the masses of the spheres relative to Σ should be m_1 and m_2 respectively both before and after the collision.

It will now be assumed that linear momentum is conserved relative to Σ as well as Σ'. Equating the total linear momentum in the $+y$ direction relative to Σ before collision to the total

momentum in the $+y$ direction relative to Σ after collision, we have

$$m_1(u_1)_y + m_2(u_2)_y = m_1(\bar{u}_1)_y + m_2(\bar{u}_2)_y$$

Substituting from eqns (5.1), (5.2), (5.3) and (5.4), since $(\bar{u}_1)_y = -(u_1)_y$ and $(\bar{u}_2)_y = -(u_2)_y$, we have

$$m_1(u_1)_y + m_2(u_2)_y = -m_1(u_1)_y - m_2(u_2)_y$$

$$2m_1(u_1)_y = -2m_2(u_2)_y$$

$$\frac{m_1u'_y\sqrt{1 - v^2/c^2}}{(1 + vu'_x/c^2)} = \frac{m_2u'_y\sqrt{1 - v^2/c^2}}{(1 - vu'_x/c^2)}$$

Hence,

$$m_1(1 - vu'_x/c^2) = m_2(1 + vu'_x/c^2) \tag{5.5}$$

From eqn (4.15), for a particle having a velocity $\mathbf{u}$ in Σ and a velocity $\mathbf{u}'$, having an x' component u'_x in Σ', we have

$$1 + \frac{vu'_x}{c^2} = \sqrt{\frac{(1 - v^2/c^2)(1 - u'^2/c^2)}{(1 - u^2/c^2)}} \tag{4.15}$$

Applying eqn (4.15) to sphere 1, since before collision the velocity of sphere 1 in Σ is $\mathbf{u}_1$, and its speed in Σ' is u' having an x' component $+u'_x$, we have

$$1 + \frac{vu'_x}{c^2} = \sqrt{\frac{(1 - v^2/c^2)(1 - u'^2/c^2)}{(1 - u_1^2/c^2)}} \tag{5.6a}$$

Similarly, since before collision the velocity of sphere 2 in Σ is $\mathbf{u}_2$, and its speed in Σ' is u' having an x' component—u'_x, from eqn (4.15) we have

$$1 - \frac{vu'_x}{c^2} = \sqrt{\frac{(1 - v^2/c^2)(1 - u'^2/c^2)}{(1 - u_2^2/c^2)}} \tag{5.6b}$$

Substituting from eqns (5.6a) and (5.6b) into eqn (5.5),

$$m_1\sqrt{1 - u_1^2/c^2} = m_2\sqrt{1 - u_2^2/c^2} \tag{5.7}$$

If the y component of linear momentum is to be conserved in Σ, the above relation must be satisfied. Eqn (5.7) must be true whatever the magnitude of u', so that it must hold for various values of u_1 and u_2. If $m_1 = m_0$ when $u_1 = 0$ and $m_2 = m_0$ when $u_2 = 0$, eqn (5.7) can always be satisfied, whatever the values of u_1 and u_2, provided

$$m_1 = \frac{m_0}{\sqrt{1 - u_1^2/c^2}} \quad \text{and} \quad m_2 = \frac{m_0}{\sqrt{1 - u_2^2/c^2}}$$

since eqn (5.7) then reduces to $m_0 = m_0$. The x component of the total linear momentum relative to Σ is also conserved in the collision. Hence, linear momentum can be conserved in both Σ and Σ', if the mass of a moving particle is redefined, such that, when it is moving with a velocity $\mathbf{u}$ relative to Σ, its mass m is equal to

$$m = \frac{m_0}{\sqrt{1 - u^2/c^2}} \tag{5.8}$$

where m_0 is the mass of the particle measured in the inertial frame in which it is at rest. Relative to Σ' the mass of the particle must be redefined as $m_0(1 - u'^2/c^2)^{-\frac{1}{2}}$ where $\mathbf{u}'$ is the velocity of the particle relative to Σ'. The quantity m_0 is called either the *rest* mass or the *proper* mass of the particle. The quantity m in eqn (5.8) is called the relativistic mass. It can be seen from eqn (5.8) that the mass of a particle is a scalar quantity. The theory developed above refers to the *inertial* mass of the particle. It must be emphasized that $\mathbf{u}$ is the velocity of the particle relative to Σ, which can be taken to be the laboratory system. The velocity $\mathbf{u}$ has nothing whatsoever to do with changing co-ordinate systems. From the requirement that the quantities appearing in the laws of mechanics must transform according to the transformations of the theory of special relativity, it was necessary to redefine the mass of a moving particle in the laboratory system. It will now be assumed that the mass of a moving particle must be defined by eqn (5.8) in the general case. This assumes that the mass of a particle is independent of the acceleration of the particle relative to the laboratory. This redefinition of mass leads one in turn to a new set of laws to replace Newton's laws of motion. These new laws will be applied in the laboratory system, and it is shown in Section 5.4 that the predictions of the new theory are in excellent agreement with experiments carried out in the laboratory.

In relativistic mechanics the momentum of a particle moving with velocity $\mathbf{u}$ is defined by the relation

$$\mathbf{p} = m\mathbf{u} = \frac{m_0\mathbf{u}}{\sqrt{1 - u^2/c^2}} \tag{5.9}$$

where $\mathbf{p}$ is the momentum, m_0 the rest mass and m is the relativistic mass of the particle. The momentum has components

$$p_x = \frac{m_0 u_x}{\sqrt{1 - u^2/c^2}}; \quad p_y = \frac{m_0 u_y}{\sqrt{1 - u^2/c^2}}; \quad p_z = \frac{m_0 u_z}{\sqrt{1 - u^2/c^2}} \tag{5.10}$$

Notice that it is the total velocity u that appears in the denominator of the components of momentum.

Before discussing the experimental verification of the variation of mass with velocity, the dynamics of a single particle is now developed, using eqns (5.8) and (5.9).

5.3. THE RELATIVISTIC DYNAMICS OF A SINGLE PARTICLE

5.3.1. *The Definition of Force*

In Newtonian mechanics the force acting on a particle can be defined as the rate of change of the momentum of the particle, under the influence of the force. Since, in Newtonian mechanics, the inertial mass of a particle is assumed to be invariant, one could write

$$\mathbf{f} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(m\mathbf{u}) = m\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = m\mathbf{a} \tag{5.11}$$

Hence, in Newtonian mechanics, the force acting on a particle could also be defined as the product of the mass of the particle and the acceleration produced by the force. If the inertial mass of a particle varies with velocity, as given by eqn (5.8), then

$$\mathbf{f} = \frac{\mathrm{d}}{\mathrm{d}t}(m\mathbf{u}) = m\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} + \mathbf{u}\frac{\mathrm{d}m}{\mathrm{d}t} \tag{5.12}$$

In addition to the term equal to mass times acceleration, there is also a term equal to $\mathbf{u}\dfrac{\mathrm{d}m}{\mathrm{d}t}$ in eqn (5.12). In the theory of special relativity, the definition of force as the rate of change of momentum is in general not equivalent to defining force as the product of mass and acceleration. In the theory of special relativity, it is chosen to define force as the rate of change of momentum. The suitability of this choice depends, of course, on whether or not the predictions of the resulting theory are in agreement with experiments. The additional term $\mathbf{u}(\mathrm{d}m/\mathrm{d}t)$ in eqn (5.12) arises from the variation of mass with velocity. If the acceleration of the particle gives rise to a change in the velocity of the particle, then there is a corresponding change in the mass of the particle. Let this change in mass be denoted by δm. Associated with this change of mass there must be a change in the momentum of the particle in the direction of motion equal to $\delta m\mathbf{u}$. To produce this change of momentum in a time δt there must be a force equal to $\delta m\mathbf{u}/\delta t$.

5.3.2. *Work and Kinetic Energy*

As in Newtonian mechanics the work done on a particle will be defined as the force $\mathbf{f}$ acting on the particle multiplied by the

distance through which the particle is displaced in the direction in which the force is acting, that is the work done in a displacement $\mathbf{dl}$ will be defined as the scalar product of $\mathbf{f}$ and $\mathbf{dl}$, that is

$$\mathrm{d}W = \mathbf{f} \cdot \mathrm{d}\mathbf{l} \tag{5.13}$$

If it is assumed that all this work goes into increasing the kinetic energy of the particle, then,

$$\mathrm{d}T = \mathbf{f} \cdot \mathrm{d}\mathbf{l} \tag{5.14}$$

where the symbol T is used to denote the kinetic energy of the particle. The rate of increase of kinetic energy is given by

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \mathbf{f} \cdot \frac{\mathrm{d}\mathbf{l}}{\mathrm{d}t} = \mathbf{f} \cdot \mathbf{u} \tag{5.15}$$

where $\mathbf{u}$ is the velocity of the particle. Now by definition

$$\mathbf{f} = \frac{\mathrm{d}}{\mathrm{d}t}(m\mathbf{u}), \text{ where } m = \frac{m_0}{\sqrt{1 - u^2/c^2}}$$

Substituting for $\mathbf{f}$ in eqn (5.15),

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(m\mathbf{u}) \cdot \mathbf{u}$$

$$= m\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} \cdot \mathbf{u} + \frac{\mathrm{d}m}{\mathrm{d}t}\mathbf{u} \cdot \mathbf{u} \tag{5.16}$$

Now

$$u_x^2 + u_y^2 + u_z^2 = u^2$$

Differentiating with respect to time,

$$2u_x\frac{\mathrm{d}u_x}{\mathrm{d}t} + 2u_y\frac{\mathrm{d}u_y}{\mathrm{d}t} + 2u_z\frac{\mathrm{d}u_z}{\mathrm{d}t} = 2u\frac{\mathrm{d}u}{\mathrm{d}t}$$

i.e.

$$\mathbf{u} \cdot \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = u\frac{\mathrm{d}u}{\mathrm{d}t}$$

Also

$$\mathbf{u} \cdot \mathbf{u} = u^2$$

and

$$\frac{\mathrm{d}m}{\mathrm{d}t} = \frac{\mathrm{d}u}{\mathrm{d}t}\frac{\mathrm{d}m}{\mathrm{d}u} = \frac{\mathrm{d}u}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{m_0}{\sqrt{1 - u^2/c^2}}\right) = \frac{\mathrm{d}u}{\mathrm{d}t}\frac{m_0 u/c^2}{(1 - u^2/c^2)^{3/2}}$$

Substituting in eqn (5.16),

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{m_0}{\sqrt{1 - u^2/c^2}}\, u \frac{\mathrm{d}u}{\mathrm{d}t} + m_0 \frac{\mathrm{d}u}{\mathrm{d}t} \frac{u^3/c^2}{(1 - u^2/c^2)^{\frac{3}{2}}}$$

$$= \frac{m_0\, u \dfrac{\mathrm{d}u}{\mathrm{d}t} (1 - u^2/c^2 + u^2/c^2)}{(1 - u^2/c^2)^{\frac{3}{2}}}$$

$$= \frac{m_0 u}{(1 - u^2/c^2)^{\frac{3}{2}}} \frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{m_0 c^2}{\sqrt{1 - u^2/c^2}}\right]$$

Integrating,

$$T = \frac{m_0 c^2}{\sqrt{1 - u^2/c^2}} + C_1$$

where C_1 is a constant of integration. If it is assumed that the kinetic energy of the particle is zero when the velocity of the particle is zero, then C_1 must be equal to $-m_0c^2$. Hence

$$T = \frac{m_0 c^2}{\sqrt{1 - u^2/c^2}} - m_0 c^2 = m_0 c^2 \left\{\frac{1}{\sqrt{1 - u^2/c^2}} - 1\right\} \tag{5.17}$$

or

$$T = mc^2 - m_0 c^2 \tag{5.18}$$

and

$$mc^2 = T + m_0 c^2 \tag{5.19}$$

Expanding the right-hand side of eqn (5.17) by the binomial theorem,

$$T = m_0 c^2 \left\{1 + \frac{1}{2}\frac{u^2}{c^2} + \frac{3}{8}\frac{u^4}{c^4} + \cdots\right\} - m_0 c^2$$

$$= \frac{1}{2} m_0 u^2 + \frac{3}{8} m_0 \frac{u^4}{c^2} + \cdots$$

If $u \ll c$ then

$$T \simeq \tfrac{1}{2} m_0 u^2$$

This is in agreement with the formula for the kinetic energy of the particle given by Newtonian mechanics. It can be seen from eqn (5.17) that, if u tends to c, then the kinetic energy T tends to infinity. Hence, according to eqn (5.17), an infinite amount of work would have to be done to accelerate a particle up to the velocity of light. This illustrates how, in the theory of special relativity, the velocity of light plays the role of a limiting velocity for a particle. It must be stressed that the theory of special relativity does not say that one cannot have velocities exceeding the velocity

of light *in vacuo*, but simply says that energy and momentum cannot be transmitted with a velocity exceeding the velocity of light *in vacuo*. As an example of a velocity exceeding the velocity of light, consider two rulers which cross each other obliquely and lie one above the other as shown in *Figure 5.2*. Let the angle between the rulers be θ. Let one of the rulers be stationary and parallel to the x axis and let the other ruler move with velocity u in the $+y$ direction as shown in *Figure 5.2*. The point of intersection of the rulers moves in the negative x direction with a velocity equal to $u \cot \theta$. If

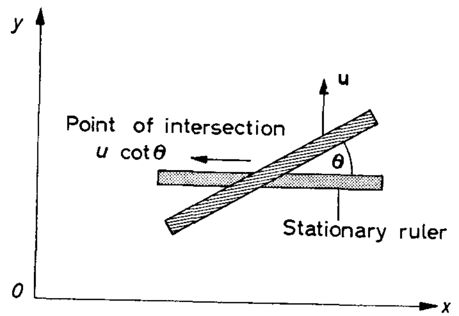


Figure 5.2. One ruler is stationary, whereas the other ruler moves with uniform velocity $\mathbf{u}$ *in the* $+y$ *direction. The point of intersection of the rulers moves with a speed* $u \cot \theta$, *which may exceed the velocity of light even though* $\mathbf{u}$ *is always less than* c

$u = 0{\cdot}8c$ and $\theta = 15$ degrees, then $u \cot \theta$ is approximately equal to $3c$. Thus the point of intersection of the rulers can move with a velocity exceeding the velocity of light. The momentum and energy of the moving ruler travel in the $+y$ direction with a velocity u, which is always less than c. Another example of a velocity greater than the velocity of light is the phase velocity of x-rays in crystals, for which the refractive index is less than unity. The energy of the waves travels with the group velocity, which is always less than or equal to c. In a medium of refractive index $n > 1$, the velocity of light is c/n. Particle velocities exceeding this are possible and have been observed. It is the velocity of light *in vacuo* which is the limiting velocity for transmitting energy and momentum.

Eqn (5.18) can be written as

$$T = mc^2 - m_0c^2 = (m - m_0)c^2 \tag{5.20}$$

showing that associated with a change in the kinetic energy of a particle there is a change in its inertial mass.

It is convenient to introduce a quantity E defined by the relation

$$E = T + m_0c^2 = mc^2 = \frac{m_0c^2}{\sqrt{1 - u^2/c^2}} \tag{5.21}$$

Thus, E is equal to the sum of the kinetic energy of the particle and the quantity m_0c^2, which is generally called the rest mass energy of a free particle. In Section 5.8 the interconvertibility of rest mass energy and other forms of energy is illustrated. The quantity E is called the total energy of a free particle or sometimes, simply the energy of the particle. The total energy E is a scalar. The symbol **E** is also used for the electric intensity vector. No confusion should arise as it will always be obvious from the text to which quantity the symbol E refers. Some writers prefer to use the symbol W for the total energy of a particle.

Now, from eqn (5.9),

$$\mathbf{p} = m\mathbf{u} = \frac{m_0\mathbf{u}}{\sqrt{1 - u^2/c^2}}$$

Hence

$$\begin{aligned} p^2c^2 + m_0^2c^4 &= \frac{m_0^2u^2c^2}{(1 - u^2/c^2)} + m_0^2c^4 \\ &= \frac{m_0^2u^2c^2 + m_0^2c^4 - m_0^2u^2c^2}{(1 - u^2/c^2)} \\ &= \frac{m_0^2c^4}{(1 - u^2/c^2)} = E^2 \end{aligned}$$

that is,

$$E^2 = p^2c^2 + m_0^2c^4 \tag{5.22}$$

or

$$E = c\sqrt{p^2 + m_0^2c^2} \tag{5.23}$$

Eqn (5.23) is an important relation which is used frequently in high energy nuclear physics to calculate the total energy of a particle, when the momentum is given and *vice versa*.

Now

$$T = E - m_0c^2$$

Hence,

$$T = c\sqrt{m_0^2c^2 + p^2} - m_0c^2 \tag{5.24}$$

Eqn (5.24) relates the kinetic energy and the momentum of a particle; this relation is also used extensively in high energy nuclear physics.

Differentiating eqn (5.23) with respect to p,

$$\frac{dE}{dp} = \frac{pc}{\sqrt{m_0^2c^2 + p^2}} = \frac{pc^2}{c\sqrt{m_0^2c^2 + p^2}} = \frac{pc^2}{E}$$

But

$$E = mc^2 \quad \text{and} \quad \mathbf{p} = m\mathbf{u}$$

Hence

$$\frac{dE}{dp} = u \tag{5.25}$$

Newton's laws are generally sufficiently accurate, except when the velocities of the particles are comparable to the velocity of light. The only available particles which have such very high velocities are high energy atomic particles, such as protons, electrons, mesons etc. The main application of the relativistic dynamics of a single particle is to the motion of such particles in electric and magnetic fields. It is assumed in relativistic electromagnetic theory that when it is not radiating, the force acting on a charge q moving with a velocity $\mathbf{u}$ in an electric field of strength $\mathbf{E}$ and a magnetic field of strength $\mathbf{B}$ is given by the Lorentz force:

$$\mathbf{f} = q\mathbf{E} + q\mathbf{u} \times \mathbf{B} \tag{5.26}$$

where $\mathbf{u} \times \mathbf{B}$ is the vector product of $\mathbf{u}$ and $\mathbf{B}$. The value of the charge q is assumed to be independent of the velocity of the particle. This is the principle of constancy of electric charge. The values of $\mathbf{E}$ and $\mathbf{B}$ are those calculated from the charge and current distributions using Maxwell's equations. The application of eqn (5.26) to the dynamics of a single particle is reviewed in Section 5.4.

5.3.3. *Units*

The equations developed in Section 5.3.2 can be applied using any consistent set of units, such as the metre-kilogram-second (M.K.S.) system of units. M.K.S. units are not always convenient in atomic physics. For example, the rest mass of the electron is $9{\cdot}11 \times 10^{-31}$ kg. It is generally more convenient in atomic and nuclear physics to work with a system of units based on the electron volt as the unit of energy.

High energy particles are generally obtained by accelerating the particles in an accelerator. It is shown in Section 5.4.1 that if a particle of charge q is accelerated through a potential difference V_0 volts, then the gain in the kinetic energy of the particle is equal to qV_0 volts. It is convenient to measure the kinetic energies of moving charged particles in terms of the potential difference through

which a particle of electronic charge would have to be accelerated in order to have the same kinetic energy as the moving particles. If a charge of 1 C is moved through a potential difference of 1 V, the work done is equal to 1 J. The work done in moving a charge of $1{\cdot}602 \times 10^{-19}$ C through a potential difference of 1 V is equal to $1{\cdot}602 \times 10^{-19}$ J. Hence

$$1\text{ eV} = 1{\cdot}602 \times 10^{-19}\text{ J} = 1{\cdot}602 \times 10^{-12}\text{ ergs} \tag{5.27}$$

For β-rays and γ-rays it is often convenient to measure energies in thousands of electron volts or in keV.

$$1\text{ keV} = 10^3\text{ eV} = 1{\cdot}602 \times 10^{-16}\text{ J} = 1{\cdot}602 \times 10^{-9}\text{ ergs}$$

For higher energy particles, it is convenient to measure energies in millions of electron volts or in MeV.

$$1\text{ MeV} = 10^6\text{ eV} = 1{\cdot}602 \times 10^{-13}\text{ J} = 1{\cdot}602 \times 10^{-6}\text{ ergs} \tag{5.28}$$

For particles of still higher energy, energies are measured in BeV or GeV, where

$$\left.\begin{array}{ll} & 1\text{ BeV} \\ \text{or} & 1\text{ GeV}\end{array}\right\} = 10^9\text{ eV} = 1{\cdot}602 \times 10^{-10}\text{ J} = 1{\cdot}602 \times 10^{-3}\text{ ergs}$$

Generally, the energy quoted refers to the kinetic energy of the particle, but at other times, particularly in β-decay work, the quoted energy often includes the rest mass energy of the particle.

Now in the equation

$$E^2 = c^2p^2 + m_0^2c^4 \tag{5.22}$$

both cp and m_0c^2 have the dimensions of energy and can be expressed in eV or MeV. For example, m_0c^2 has the value 0·51098 MeV for an electron. Some texts simply quote the masses of particles in MeV, implying that this stands for the quantity m_0c^2.

As an example of the appropriate units for momentum, the momentum of an electron of kinetic energy 2 MeV will be calculated. Since $m_0c^2 = 0{\cdot}511$ MeV, E the total energy of the electron is 2·511 MeV. Using eqn (5.22) and working in MeV $(2{\cdot}511)^2 = c^2p^2 + (0{\cdot}511)^2$

$$cp = (6{\cdot}305 - 0{\cdot}261)^{\frac{1}{2}} = 2{\cdot}46\text{ MeV}$$

Momenta are generally written in the form

$$p = 2{\cdot}46\text{ MeV}/c$$

and it is said that momenta are measured in units of eV/c or MeV/c.

As a further example of the use of eqn (5.22) the rest mass of a charged particle which has a momentum of 300 MeV/c when its

kinetic energy is 190 MeV will be calculated. From eqn (5.22)

$$E^2 = (T + m_0c^2)^2 = c^2p^2 + m_0^2c^4$$

Rearranging,

$$m_0c^2 = (c^2p^2 - T^2)/2T = (300^2 - 190^2)/2 \times 190$$
$$= 142 \text{ MeV}$$

or

$$m_0 = 142 \text{ MeV}/c^2$$

This is about 277 times the rest mass of an electron, and the particle is probably a π-meson.

In nuclear physics the basic unit of mass was, until recently, the atomic mass unit or a.m.u. This was defined as $\frac{1}{16}$ of the rest mass of the common isotope of oxygen. One a.m.u. was equal to $1{\cdot}6598 \times 10^{-27}$ kg. According to the relation $E = m_0c^2$, this rest mass energy was equivalent to

$$E = m_0c^2 = 1{\cdot}66 \times 10^{-27} \times (3 \times 10^8)^2 = 1{\cdot}491 \times 10^{-10}\,\text{J}$$

Since 1 MeV is equivalent to $1{\cdot}602 \times 10^{-13}$ J,

$$1 \text{ a.m.u.} \equiv \frac{1{\cdot}491 \times 10^{-10}}{1{\cdot}602 \times 10^{-13}} \equiv 931 \text{ MeV}/c^2$$

Hence

$$1 \text{ a.m.u.} \equiv (1/16)^{16}\text{O} \equiv 1{\cdot}6598 \times 10^{-27}\,\text{kg} \equiv 1{\cdot}491 \times 10^{-10}\,\text{J}/c^2$$
$$\equiv 931 \text{ MeV}/c^2 \qquad (5.29)$$

The energy equivalent to the rest mass of the electron and the proton are 0·511 MeV and 937 MeV respectively. The Eleventh General Congress on Weights and Measures suggested that a new unit of atomic mass should be adopted, namely a unit which is equal to $\frac{1}{12}$ of the rest mass of the ^{12}C isotope. One new a.m.u. = $\frac{1}{12}$ mass $^{12}\text{C} = 1{\cdot}6603 \times 10^{-27}$ kg. The new scale will gradually come into common usage.

5.3.4. *Trigonometrical Methods*

Mathematical calculations can often be simplified by using the trigonometrical substitution

$$\sin\theta = u/c \qquad (5.30)$$

where u is the velocity of the particle. We then have

$$m = \frac{m_0}{\sqrt{1 - \sin^2\theta}} = m_0 \sec\theta \qquad (5.31)$$

The increase of mass with velocity is obtained by plotting m/m_0 against u/c, that is, sec θ against sin θ as shown in *Figure 5.3*(*a*). If the factor $1/(1 - u^2/c^2)^{\frac{1}{2}} = \sec\theta$ is given, then $u/c = \sin^{-1}\theta$ can be determined from trigonometrical tables, and *vice versa.*

The total energy of a particle is given by

$$E = mc^2 = m_0c^2 \sec\theta \tag{5.32}$$

The kinetic energy is given by

$$T = E - m_0c^2 = m_0c^2(\sec\theta - 1) \tag{5.33}$$

Now,

$$c^2p^2 = E^2 - m_0^2c^4 = m_0^2c^4(\sec^2\theta - 1) = m_0^2c^4\tan^2\theta$$

or

$$cp = m_0c^2 \tan\theta \tag{5.34}$$

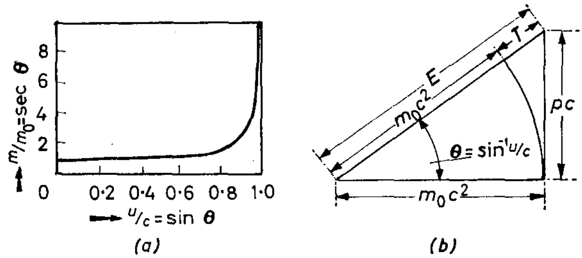


Figure 5.3. (*a*) sec θ *is plotted against* sin θ. *This is the same as plotting the ratio of the relativistic mass m of a particle to its rest mass* m_0 *against the ratio of the velocity of the particle to the velocity of light.* (*b*) *The relations between E, T, p and* m_0 *are shown graphically*

The relation $E^2 = p^2c^2 + m_0^2c^4$ can also be represented graphically. The quantities E, pc, and m_0c^2 form the sides of a right-angled triangle as shown in *Figure 5.3*(*b*). We have

$$\sin\theta = \frac{pc}{E} = \frac{muc}{mc^2} = u/c$$

It is left as an exercise for the reader to show that eqns (5.32), (5.33) and (5.34) can be developed from *Figure 5.3*(*b*) using trigonometrical methods.

The kinetic energy calculated from the equation $T = \frac{1}{2}m_0u^2$ deviates from the relativistic value by 1 per cent for an electron of 3·2 keV, a μ-meson of 0·66 MeV and a proton of 6·1 MeV. It can be seen that relativistic effects become important for electrons when they are accelerated through a potential difference of a few kilovolts. If the same value of velocity is used in both cases, the kinetic

energy calculated using the non-relativistic formula is only 50 per cent of the value predicted by the relativistic formula for kinetic energy when electrons, μ-mesons and protons have relativistic kinetic energies of 0·31 MeV, 64 MeV and 590 MeV respectively.

5.4. SOME APPLICATIONS OF THE RELATIVISTIC DYNAMICS OF A SINGLE PARTICLE

5.4.1. *Motion in an Electric Field (No Magnetic Field)*

It will be assumed that there is a uniform electric field of strength E V/m in the positive x direction between the plates of a large parallel plate capacitor. If the capacitor is charged to a potential difference of V_0 volts then the electric field is given by

$$E = E_x = V_0/d \tag{5.35}$$

where d is the separation of the plates measured in metres. If it is assumed that the value of the total electric charge on a particle is independent of the velocity of the charge, then the electric force on a charge $+q$ coulombs in the electric field $\mathbf{E}$ is given by

$$\mathbf{f} = q\mathbf{E} \text{ newtons}$$

whatever the velocity of the charge. In general one has

$$\mathbf{f} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{m_0\mathbf{u}}{\sqrt{1 - u^2/c^2}}\right) = q\mathbf{E}$$

For motion in the x direction,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{u_x}{\sqrt{1 - u_x^2/c^2}}\right) = q\,\frac{E}{m_0}$$

Integrating,

$$\frac{u_x}{\sqrt{1 - u_x^2/c^2}} = \frac{qE}{m_0}t + C_1$$

where C_1 is a constant of integration. If it is assumed that $u_x = 0$ at $t = 0$, then $C_1 = 0$. Rearranging one has

$$u_x^2 = \left(1 - \frac{u_x^2}{c^2}\right)\frac{q^2E^2t^2}{m_0^2}$$

that is

$$u_x^2 = \frac{q^2E^2t^2/m_0^2}{[1 + q^2E^2t^2/m_0^2c^2]}$$

or

$$u_x = \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{qEt}{m_0[1 + q^2E^2t^2/m_0^2c^2]^{\frac{1}{2}}} = \frac{c}{\{m_0^2c^2/q^2E^2t^2 + 1\}^{\frac{1}{2}}} \tag{5.36}$$

As $t \to \infty$, the speed of the charged particle approaches the speed of light. An experimental check of this result is described in Section 5.4.2. Eqn (5.36) gives the velocity of the charge at any time t. Integrating eqn (5.36)

$$x = \int \frac{qEt\,\mathrm{d}t}{m_0[1 + q^2E^2t^2/m_0^2c^2]^{\frac{1}{2}}} = \frac{m_0c^2}{qE}\left[1 + \frac{q^2E^2t^2}{m_0^2c^2}\right]^{\frac{1}{2}} + C_2$$

where C_2 is a constant of integration. If $x = 0$ at $t = 0$, then

$$0 = \frac{m_0c^2}{qE} + C_2 \quad \text{or} \quad C_2 = -\frac{m_0c^2}{qE}$$

Hence,

$$x = \frac{m_0c^2}{qE}\left[\left\{1 + \frac{q^2E^2t^2}{m_0^2c^2}\right\}^{\frac{1}{2}} - 1\right] \tag{5.37}$$

This expression gives the distance the charge moves in a time t. Eqn (5.37) can be rewritten as

$$\left(x + \frac{m_0c^2}{qE}\right)^2 - c^2t^2 = \frac{m_0^2c^4}{q^2E^2} \tag{5.38}$$

This is the equation of a hyperbola; for this reason this type of motion is sometimes called hyperbolic motion, the graph of x against t being a hyperbola.

If qEt/m_0 is very much less than c, then eqn (5.36) becomes

$$u_x = \left(\frac{qE}{m_0}\right)t$$

Now, qE/m_0 is the non-relativistic value for the acceleration, so that eqn (5.36) reduces to the classical case when both the acceleration and the time are small. Expanding eqn (5.37) by the binomial theorem one has

$$x = \frac{m_0c^2}{qE}\left[1 + \tfrac{1}{2}q^2\frac{E^2t^2}{m_0^2c^2} + \cdots - 1\right]$$

If $qEt/m_0 \ll c$, this reduces to

$$x = \tfrac{1}{2}\left(\frac{qE}{m_0}\right)t^2$$

This is again in agreement with the non-relativistic value.

The total work done on the charge by the electric field is given by

$$\text{work done} = \int F_x\,\mathrm{d}x = \int qE_x\,\mathrm{d}x$$

But $E_x = -\dfrac{\partial\phi}{\partial x}$, where ϕ is the electrostatic potential. Hence, the work done on the charge is equal to

$$-q\int_0^x \frac{\delta\phi}{\delta x}\,\mathrm{d}x = -q\int_{\phi=V_0}^{\phi=0}\mathrm{d}\phi = qV_0 \text{ joules}$$

where V_0 is the potential difference through which the charge has moved. Equating the work done on the charge to the gain in the kinetic energy of the charge,

$$T = qV_0 \tag{5.39}$$

5.4.2. *Speed and Kinetic Energy of Relativistic Electrons*

The experimental arrangement used by Bertozzi[2] (1964) to measure the speeds of high energy electrons, accelerated by an electric field, is shown in *Figure 5.4*. Bursts of electrons lasting about 3×10^{-9} sec, were accelerated, by the electric field in a Van de Graaff electrostatic generator, up to energies of 1·5 MeV. The electrons were then injected into an electron linear accelerator. Between the Van de Graaff and the linear accelerator there was a short insulated metal tube which collected some of the electrons in

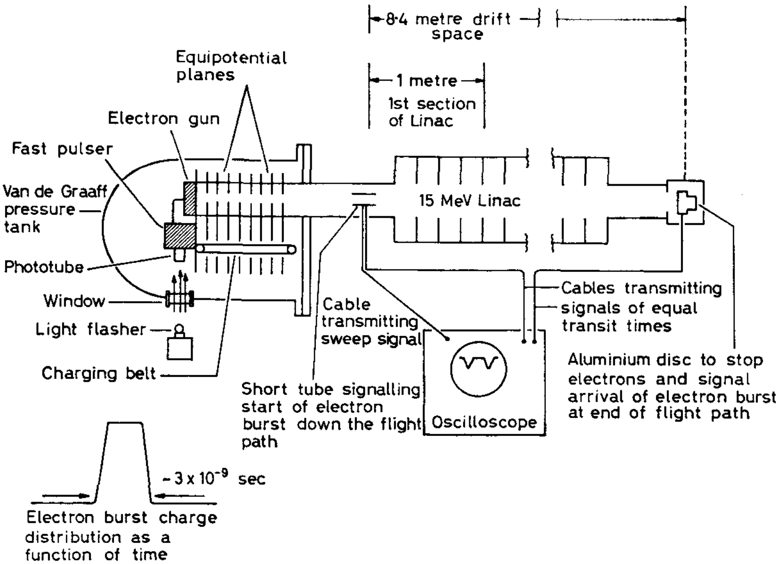
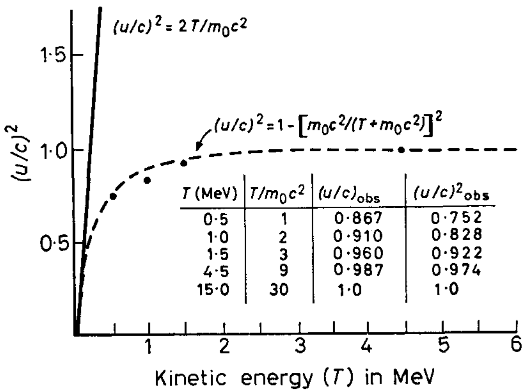


Figure 5.4. Experimental arrangement used by Bertozzi[2] to measure the time of flight of a burst of electrons accelerated by a Van de Graaff generator
(By courtesy of *Am. J. Physics*)

each burst passing through it, giving rise to a voltage pulse which signalled the start of the electron burst down the flight path, which in this experiment was the metal pipe forming the linear accelerator. At the end of the linear accelerator, after a flight path of 8·4 m, the electrons were stopped in an aluminium disk. The charging of this disk produced a second electrical signal. The two electrical signals, one from the beginning and one from the end of the flight path, were transmitted along two separate cables of *equal* length, and therefore of equal transit times, to a cathode ray oscilloscope. The calibrated sweep of the oscilloscope was triggered earlier by another signal transmitted from the beginning of the flight path via a third shorter cable. The separation on the oscilloscope trace of the two pulses transmitted by the equal length cables gave the time of flight of the electrons over a distance of 8·4 m.

Kinetic energies up to 1·5 MeV were obtained from the Van de Graaff without the linear accelerator turned on. To a good approximation the linear accelerator served only as an evacuated metal enclosure with no accelerating electric fields. If t is the time electrons take to travel a distance of 8·4 m, their speed is $8{\cdot}4/t$. The results obtained are tabulated and shown graphically in *Figure 5.5.* The speed measurement at 4·5 MeV was performed



T (MeV)	T/m_0c^2	$(u/c)_{obs}$	$(u/c)^2_{obs}$
0·5	1	0·867	0·752
1·0	2	0·910	0·828
1·5	3	0·960	0·922
4·5	9	0·987	0·974
15·0	30	1·0	1·0

Figure 5.5. Experimental results obtained by Bertozzi[2]. *The solid curve represents the prediction of* $(u/c)^2$ *according to Newtonian mechanics,* $(u/c)^2 = 2T/m_0c^2$. *The dashed curve represents the prediction of the theory of special relativity:* $(u/c)^2 = 1 - [m_0c^2/(T + m_0c^2)]^2$ *where* m_0 *is the rest mass of the electron and* $c = 3 \times 10^8$ m/sec *is the speed of light in a vacuum. The solid circles are the experimental points. They agree with the theory of special relativity, and show that* c *is the maximum speed for electrons*

(By courtesy of *Am. J. Physics*)

with the Van de Graaff set at 1·5 MeV and with the first section of the linear accelerator (about 1 m long) being used to supply the additional 3 MeV. In this case the speed of the electrons varied over the first metre of the 8·4 m flight path. The results presented in *Figure 5.5* show that, as the kinetic energy of the electrons increases, the speed of the electrons approaches the speed of light. The speed of electrons only increased by 14 per cent between kinetic energies of 0·5 and 4·5 MeV. In the 15 MeV run, the electron energy was increasing continually over most of the flight path. However, the time of flight was almost the same as for 4·5 MeV electrons showing that the speed of the electrons did not increase appreciably between 4·5 and 15 MeV. The results of Bertozzi show conclusively that the speed of light is the limiting speed for electrons. If space is isotropic, the limiting speed of electrons should have the same value in all directions of space.

From eqn (5.19)

$$mc^2 = \frac{m_0c^2}{\sqrt{1 - u^2/c^2}} = T + m_0c^2$$

Rearranging,

$$\frac{u^2}{c^2} = 1 - \left\{\frac{m_0c^2}{(T + m_0c^2)}\right\}^2 \tag{5.40}$$

Eqn (5.40) is plotted in *Figure 5.5.* The kinetic energy T was assumed to be equal to $\int q\mathbf{E} \cdot d\mathbf{l}$ where $\mathbf{E}$ is the electric field, q the charge on an electron and $d\mathbf{l}$ is an element of length of the electron path. It can be seen that the experimental results of Bertozzi are in agreement with the variation of speed with kinetic energy predicted by the theory of special relativity.

Using a thermocouple, Bertozzi measured the temperature rise and the heat generated in the aluminium disk by the electrons when they are stopped at the end of the flight path. From the charge collected by the aluminium disk, the total number of electrons was estimated. Assuming that all the kinetic energy of the electrons goes into heat, the kinetic energy of the electrons was determined. Energy measurements were performed at the accelerator settings of 1·5 and 4·5 MeV. The corresponding kinetic energy values obtained by the direct thermal technique were 1·6 and 4·8 MeV respectively. These values agreed within the experimental error of 10 per cent with the values for kinetic energy calculated from $\int q\mathbf{E} \cdot d\mathbf{l}$. They showed that, though the speeds of electrons increased only slightly between the 1·5 and 4·5 MeV

settings of the accelerator, the kinetic energies of the electrons increased by a factor of 3.

If the laws of high-speed mechanics are to obey the principle of relativity, the analysis of Section 5.4.1 and the experiment of Bertozzi[2] should be valid in all inertial frames. For example, consider a rocket moving with a velocity $+\mathbf{v}$ relative to the earth. If the laws of high-speed mechanics obey the principle of relativity, according to the analysis of Section 5.4.1, when it is accelerated in an electric field, an electron starting from rest relative to the rocket should tend to a limiting speed relative to the rocket. If the electric field, relative to the earth is parallel to $-\mathbf{v}$ such that the acceleration of the electron is parallel to $+\mathbf{v}$, the analysis of Section 5.4.1 should be valid relative to the earth also. The speed of the electron should tend to a limiting value c, relative to the earth. This limiting speed, or ultimate speed, relative to the earth, should be independent of the speed of the rocket, relative to the earth. Hence, the speeds of accelerated electrons should tend to a limiting value in all inertial frames. If all inertial frames are equivalent, the limiting speed for accelerated electrons should have the same numerical value in all inertial frames, provided the same fundamental units of length and time are used. For example, if the numerical value of the limiting speed of accelerated electrons, measured in the rocket moving relative to the earth, depended on $\mathbf{v}$ the speed of the rocket relative to the earth, one could carry out experiments to find out how the limiting speed of accelerated electrons relative to the rocket varied with the speed of the rocket relative to the earth. Thereafter, by measuring the limiting speed of accelerated electrons inside a moving rocket, one could estimate the speed of the rocket relative to the earth, without looking at anything external to the rocket. This is contrary to the assumption that, in inertial reference frames, the laws of mechanics obey the principle of relativity. Therefore, if the laws of high-speed mechanics are to obey the principle of relativity, we conclude that the limiting speed of accelerated electrons is the same, and equal to the terrestrial value of $c = 3{\cdot}0 \times 10^8$ m/s, in all inertial reference frames. This result will be called the *principle of the constancy of the limiting speed of particles*. This principle together with the principle of relativity can be used to develop the Lorentz transformations in the way outlined in Appendix 3(b). By treating light as photons travelling at the limiting speed, it will be illustrated in Section 5.8.6 that the principle of the constancy of the speed of light is a special case of the principle of the constancy of the limiting speed of particles.

5.4.3. *Motion in a Magnetic Field (No Electric Field)*

The force on a particle of charge q coulombs moving with velocity $\mathbf{u}$ m/sec in a magnetic field of strength $\mathbf{B}$ Wb/m^2 is equal to

$$\mathbf{f} = q\mathbf{u} \times \mathbf{B} \text{ newtons}$$

where $\mathbf{u} \times \mathbf{B}$ is the vector product of $\mathbf{u}$ and $\mathbf{B}$. This force is perpendicular to both $\mathbf{u}$ and $\mathbf{B}$; consequently the scalar product $\mathbf{f} \cdot \mathbf{u}$ is zero so that eqn (5.15) becomes

$$\mathbf{f} \cdot \mathbf{u} = \frac{\mathrm{d}T}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(m - m_0)c^2 = c^2 \frac{\mathrm{d}m}{\mathrm{d}t} = 0$$

Thus the kinetic energy and relativistic mass of a charged particle moving in a magnetic field are constant. Hence the speed and momentum of the charged particle are also constant, and

$$\mathbf{f} = \frac{\mathrm{d}}{\mathrm{d}t}(m\mathbf{u}) = \frac{m\mathrm{d}\mathbf{u}}{\mathrm{d}t} + \frac{\mathbf{u}\mathrm{d}m}{\mathrm{d}t} = \frac{m\mathrm{d}\mathbf{u}}{\mathrm{d}t} = m\mathbf{a}$$

so that

$$\mathbf{f} = m\mathbf{a} = m\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \frac{m_0}{\sqrt{1 - u^2/c^2}} \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = q\mathbf{u} \times \mathbf{B} \tag{5.41}$$

The acceleration is in the same direction as the force in this case. The equation of motion is exactly the same as the classical case except that m_0 is replaced by $m_0/(1 - u^2/c^2)^{\frac{1}{2}}$ which remains constant.

We shall start by considering motion in a uniform magnetic field. Let the initial velocity of the particle be perpendicular to the direction of the magnetic field as shown in *Figures 5.6(a)* and *(b)*. Let the magnetic field be in the negative z direction. The acceleration is always perpendicular to $\mathbf{u}$ so that the velocity of the particle is constant in magnitude. In a uniform magnetic field the acceleration is equal to quB/m and is a constant so that the charged particle moves in a circle as shown in *Figures 5.6(a)* and *(b)*. If the radius of the circular motion is ρ, then the centripetal acceleration is equal to u^2/ρ as illustrated in *Figures 5.6(b)* and *(c)*.

Substituting in eqn (5.41)

$$f = m\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{mu^2}{\rho} = quB$$

or

$$mu = q(B\rho) = p \tag{5.42}$$

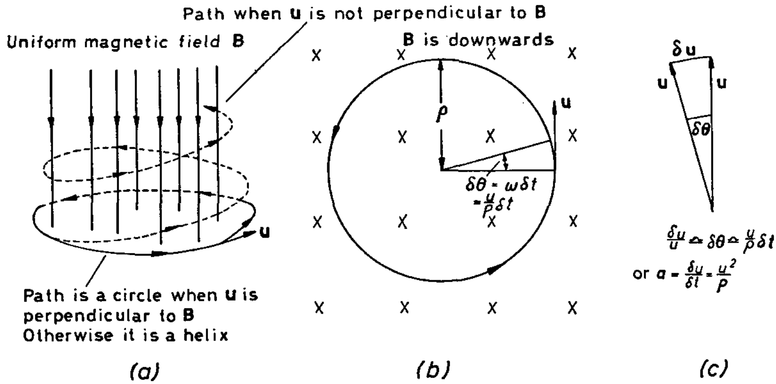


Figure 5.6. The motion of a positive charged particle in a uniform magnetic field is a circle, if the initial velocity of the particle is perpendicular to the magnetic field **B**. *When* **u** *is not perpendicular to* **B** *the path is a helix*

If a charged particle moves in a circle of radius ρ in a uniform magnetic field B then the time for one complete revolution is equal to

$$2\pi\rho/u = \frac{2\pi\rho m}{p} = \frac{2\pi\rho m}{qB\rho} = \frac{2\pi m}{qB} \tag{5.43}$$

If the variation of the mass of the particle with velocity can be neglected, this time is independent of the velocity of the particle. According to eqn (5.42), the radius of the orbit is proportional to the momentum of the particle.

By measuring the radius of curvature of the orbit of a charged particle moving in a direction perpendicular to a uniform magnetic field, the momentum of the particle can be determined using eqn (5.42). The greater the momentum, the larger the radius of curvature. The quantity $B\rho$ is called the magnetic rigidity. Consider a particle of charge Zq, where $q = 1{\cdot}602 \times 10^{-19}$ C is the electronic charge. If B is in Wb/m² and ρ is in metres, then from eqn (5.42)

$$pc = qB\rho c = Z \times 1{\cdot}602 \times 10^{-19} B\rho \times 3 \times 10^{8}\,\mathrm{J}$$

But

$$1\mathrm{J} = 1/1{\cdot}602 \times 10^{-13}\,\mathrm{MeV}$$

Hence

$$p = 300 ZB\rho\ \mathrm{MeV}/c \tag{5.44}$$

If B is in gauss and ρ is in centimetres

$$p = 300 ZB\rho\ \mathrm{eV}/c \tag{5.45}$$

As an example consider a charged particle of charge equal to an electron moving in a direction perpendicular to a uniform magnetic field of 1 Wb/m². If the radius of curvature of its orbit is 105 cm, calculate its momentum. If its velocity is $0{\cdot}95c$, find its rest mass and kinetic energy.

From eqn (5.44)

$$p = 300\,B\rho = 315\ \mathrm{MeV}/c$$

Now

$$cp = \frac{m_0uc}{\sqrt{1-u^2/c^2}} = \frac{0{\cdot}95m_0c^2}{\sqrt{1-0{\cdot}95^2}} = 3{\cdot}202 \times 0{\cdot}95m_0c^2$$

Hence

$$3{\cdot}202 \times 0{\cdot}95m_0c^2 = 315\ \mathrm{MeV}$$

or

$$m_0c^2 = 103{\cdot}5\ \mathrm{MeV}$$

This corresponds to a negative particle of mass $\sim$203 times the electron rest mass. It is therefore probably a μ^--meson. Now

$$E^2 = c^2p^2 + m_0^2c^4 = 315^2 + 103{\cdot}5^2$$

$$E = 331{\cdot}5\ \mathrm{MeV}$$

$$T = E - m_0c^2 = 228\ \mathrm{MeV}$$

It is left as an exercise for the reader to repeat the calculation using the trigonometrical formulae developed in Section 5.3.4.

If the velocity of the charged particle is not perpendicular to the uniform magnetic field, then the velocity of the particle can be resolved into two components, one component $\mathbf{u}_\parallel$ parallel to the magnetic field and the other component $\mathbf{u}_\perp$ perpendicular to the magnetic field. If there is no electric field present, $\mathbf{u}_\parallel$ is unchanged in magnitude and direction. This velocity must be compounded with the circular motion perpendicular to the magnetic field. The resulting motion is a helix.

The motion of ions in a non-uniform magnetic field can also be calculated from the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{m_0\mathbf{u}}{\sqrt{1-u^2/c^2}}\right) = q\mathbf{u} \times \mathbf{B} \tag{5.46}$$

When the magnetic field is not uniform and especially when both electric and magnetic fields are present, it is generally more convenient to use Lagrange's equations and start from the relativistic Lagrangian. This method is discussed by Rosser[3] Section 9.6 where it is shown that, if the Lagrangian for a charged particle of velocity

u is defined as

$$L = m_0c^2(1 - \sqrt{1 - u^2/c^2}) + q(\mathbf{u} \cdot \mathbf{A}) - q\phi \tag{5.47}$$

where ϕ is the scalar potential and $\mathbf{A}$ is the vector potential of the electromagnetic field, then Lagrange's equations give the correct relativistic equation of motion.

It is shown by Rosser[3] (Section 9.7) that if the Hamiltonian is defined as

$$H = q\phi + c\{m_0^2c^2 + (\mathbf{p} - q\mathbf{A})^2\}^{\frac{1}{2}} \tag{5.48}$$

then Hamilton's equations give the correct relativistic equation of motion for a charged particle.

5.4.4. *Experimental Verification of the Variation of Mass with Velocity*

The ratio e/m for electrons was measured by Thomson in 1897 using cathode rays. In 1901 it was shown qualitatively by

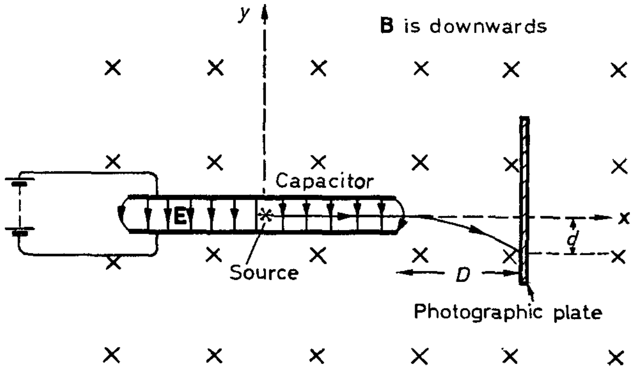


Figure 5.7. Simplified diagram of Bucherer's experiment. The capacitor acts as a velocity selector. After emerging from between the plates of the capacitor, the β-rays are deflected in a magnetic field and detected by a photographic plate

Kaufmann, using the parabola method, that the value of e/m depended on the velocity of β-rays. In 1908 Bucherer[4] carried out more accurate measurements of e/m for β-rays. The principle of Bucherer's experiment is illustrated in *Figure 5.7*; β-rays emitted from a radium source passed between the plates of a large capacitor. The diameters of the plates of the capacitor were very much bigger than the distance between the plates. The whole apparatus was placed inside a vacuum. A potential difference was applied to the plates of the capacitor so as to produce an electric field in the negative y direction as shown in *Figure 5.7*. The electric force on a

negative electron was equal to Ee and acted in the positive y direction. A magnetic field was applied in the negative z direction, that is, perpendicular to the paper away from the reader, in *Figure 5.7*. The magnetic force on an electron moving along the positive x axis was equal to euB, and was in the negative y direction. If the electric and magnetic forces acting on such an electron were not equal, then an electron moving initially along the x axis would be deflected in either the positive or negative y direction and would not emerge from between the plates of the capacitor. If the electric and magnetic forces were exactly equal, the electron would not be deflected. It is only in this case that an electron would emerge from between the plates of the capacitor, if the electron were moving initially along the x axis. For this case

$$euB = eE$$

or

$$u = E/B \tag{5.49}$$

The capacitor acted as a velocity selector in Bucherer's experiment. There was no electric field outside the capacitor, so that the electrons, which emerged from the velocity selector, moved in circular orbits in the magnetic field before striking the photographic plate, as shown in *Figure 5.7*. If ρ is the radius of the circular path outside the capacitor plates,

$$d(2\rho - d) = D^2$$

or

$$\rho = \frac{D^2 + d^2}{2d} \tag{5.50}$$

where the quantities d and D are as shown in *Figure 5.7*. From eqn (5.42) for motion in a magnetic field one has

$$mu = qB\rho \quad \text{or} \quad \rho = \frac{mu}{Be}$$

Thus

$$\frac{D^2 + d^2}{2d} = \frac{mu}{Be}$$

But from eqn (5.49), $u = E/B$, so that

$$\frac{e}{m} = \frac{2d}{(D^2 + d^2)} \frac{E}{B^2} \tag{5.51}$$

If all the lengths are measured in metres, E in volts per metre and B in webers per metre^2, then e/m is in coulombs per kilogram. Bucherer

reversed the electric and magnetic fields and obtained a second spot on the photographic plate. Bucherer took d to be equal to half the distance between the two spots. The experiment was repeated for electrons having different velocities. Some of the results obtained by Bucherer are given in Table 5.1. It can be seen that the values of $e/m = e(1 - u^2/c^2)^{\frac{1}{2}}/m_0$ varied with the velocity of the β-rays; the velocities of the β-rays were calculated using eqn (5.49). The calculated values of e/m_0 were constant. Similar experiments have

Table 5.1. Bucherer's results for e/m for β-rays

u/c	$e/m = \dfrac{e\sqrt{1 - u^2/c^2}}{m_0}$	e/m_0
0·3173	$1{\cdot}661 \times 10^{11}$ C/kg	$1{\cdot}752 \times 10^{11}$ C/kg
0·3787	1·630	1·761
0·4281	1·590	1·760
0·5154	1·511	1·763
0·6870	1·283	1·767

been performed many times since, and the results obtained have always been consistent with the equation

$$m = \frac{m_0}{\sqrt{1 - u^2/c^2}} \tag{5.52}$$

It is generally accepted that these experiments confirm eqn (5.52) for the variation of mass with velocity. However, other postulates were used in the theory of the experiment; for example, it was assumed that the force on the moving charge was given by the Lorentz force and that the electric charge of the particle was independent of its velocity (the principle of constant electric charge). It is preferable to think of the results of these experiments as one illustration of the correctness of the relativistic theory for the motion of ions in electric and magnetic fields taken as a whole, rather than as a confirmation of one particular part of the theory. There is, however, independent evidence in favour of the principle of constant charge. If the magnitude of the total charge on a particle did vary with the velocity of the particle, for example in a way similar to the variation of mass with velocity, then hydrogen atoms and molecules would not be electrically neutral, since, on the average, the electrons in hydrogen atoms and molecules have higher velocities, relative to the laboratory, than the protons. King[5] (1960) showed that the charges on the electrons and the protons in hydrogen molecules are numerically equal to within 1 part in 10^{20}. King also showed that the positive protons in helium nuclei

cancelled the charge of the orbital electrons with nearly the same experimental accuracy. The principle of constant electric charge is implicit in Maxwell's equations. (Rosser[3], Section 3.3.)

5.4.5. *The Acceleration of Charged Particles to High Energies*

This account is not intended to be an exhaustive account of particle accelerators, but merely a brief outline of some of the

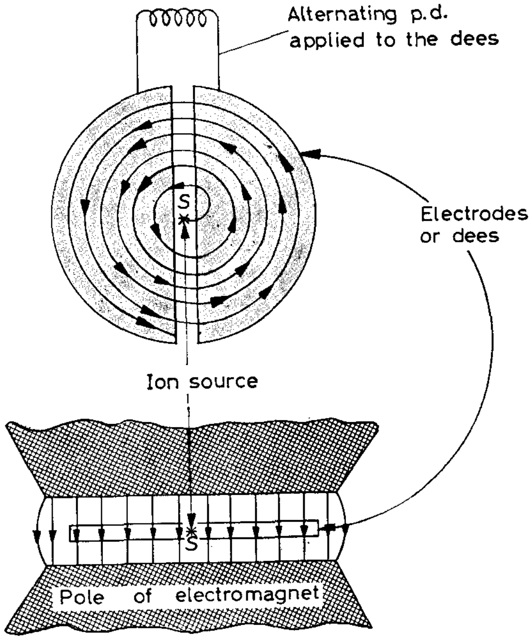


Figure 5.8. A simplified diagram of the cyclotron. The path of a typical ion is shown

physical principles underlying the acceleration of particles in the energy range where the variation of mass with velocity becomes important. For a detailed account of accelerators the reader is referred to McMillan[6] where references to other works are given.

The cyclotron is shown in schematic form in *Figure 5.8.* It consists of two flat hollow conducting semi-circular boxes (which are generally called dees). The two dees are placed between the poles of an electromagnet and are inside a high vacuum. Ions are produced from the source S which is in the centre between the two dees. A potential difference is applied between the dees such that there

is an electric field in the gap between them. The ions are accelerated from the ion source towards one of the dees. As the dees are hollow conductors the electric field inside them is negligible, so that once they are inside one of the dees, the velocities of the ions remain constant and the ions move in circular orbits inside the dees. After going through a semi-circle, the ions reach the gap between the two dees. If by this time the potential difference and the electric field between the dees are reversed in direction, the ions are accelerated again and move in a semi-circle of larger radius inside the other dee before reaching the gap again. According to eqn (5.43) the time for one complete revolution in a uniform magnetic field is equal to $2\pi\rho/u = 2\pi m/qB$, so that the time for half a revolution is equal to $\pi m/Bq$. This time is independent of the energy of the particles, provided the variation of the mass of the particles with velocity can be neglected. If the electric field between the dees reverses in direction at regular intervals of $\pi m/Bq$, then the ions will be accelerated each time they cross the gap between the dees gaining energy each time. The radius of the orbit of the ions is given by

$$\rho = \frac{mu}{qB} \tag{5.42}$$

As the velocity of the ions increases, then the ions go in semi-circles of ever increasing diameter. The alternating potential difference can be obtained from a high frequency oscillator. The condition for resonance is

$$n = \frac{qB}{2\pi m} = \frac{qB\sqrt{1 - u^2/c^2}}{2\pi m_0} \tag{5.53}$$

where n is the frequency of the oscillator.

Provided the variation of mass with velocity can be neglected the time of revolution of the ions in a uniform magnetic field remains constant and the ions can be accelerated by a fixed frequency oscillation, of frequency given by eqn (5.53). For protons, if $n = 10^7$ c/sec then the magnetic field must be 0·65 Wb/m^2 (6,500 G).

At high energies the mass of the ions increases with velocity and the time of revolution increases. If the frequency n remains fixed, the ions will get out of phase. The variation of mass with velocity is small enough for protons below 40 MeV for them to be accelerated without getting completely out of phase. It is not possible to accelerate electrons beyond $\sim$50 keV before they get out of phase. In principle the loss of resonance can be overcome in two ways.

As the energy of the ions increases the relativistic mass increases and the resonant frequency goes down. If the frequency of the applied potential difference is modulated appropriately, then it should be possible to accelerate ions to higher energies. One of the first frequency-modulated cyclotrons (or synchrocyclotrons) was the 184-inch machine developed at the University of California. This produced protons of energies up to 350 MeV, deuterons of energies up to 195 MeV and α-particles of energies up to 390 MeV. For protons the frequency had to be changed from 22·9 Mc/sec at the instant of injection to 15·8 Mc/sec when the protons reached the outside of the dees. The corresponding frequencies for deuterons and α-particles were 11·5 and 9·8 Mc/sec. The synchrocyclotron is not suitable for accelerating electrons as the rate of modulation required would be too high.

The other method of overcoming loss of resonance is to increase the magnetic field, after the ions have been injected, at such a rate that the ratio $B/m = B(1 - u^2/c^2)^{\frac{1}{2}}/m_0$ remains constant. The frequency need not then be changed. This method has been applied successfully to the acceleration of electrons in the electron synchrotron. The electrons are injected at energies of a few tens of keV and are accelerated initially by betatron action up to a few MeV, when the synchrotron action takes over and the electrons are finally accelerated up to 300–400 MeV. One of the advantages of the electron synchrotron is that, at 2 MeV, electrons move with practically the speed of light ($\sim$0·98 c). If B/m is a constant, the radii of the orbits given by eqn (5.42) do not increase very much as the energies of the electrons increase, since above 2 MeV the velocities of the electrons remain practically constant. For this reason, one need only use a doughnut-shaped vacuum chamber and the magnetic field need only extend over the dimensions of this vacuum chamber. Protons do not approach the speed of light until they have an energy of a few GeV; for example, protons do not reach the same speed as 2 MeV electrons until their energy is $\sim$4 GeV. If m/B were kept constant in a proton synchrotron, such that a constant frequency could be used, then the radii of the proton orbits would increase substantially as the protons were accelerated from a few MeV up to a few GeV. This would necessitate building an enormous magnet. In order to keep the protons in orbits of fixed radii both the frequency of the potential difference and the strength of the magnetic field are varied. The C.E.R.N. proton synchrotron can accelerate protons up to an energy of 25 GeV.

When electrons move in circular orbits they emit photons (synchrotron radiation) due to their centripetal acceleration, and

this becomes very important at energies above 300 MeV. Electrons have been accelerated to energies greater than 600 MeV in the Stanford linear electron accelerator. In this accelerator the electrons are accelerated by the electric vector of a travelling electromagnetic wave. The linear electron accelerator is designed such that the speed of the electromagnetic wave increases from the speed of the electrons at injection towards the speed of light. When the electron energy is greater than 600 MeV its relativistic mass is greater than 1,200 times the electron rest mass.

The design of high energy accelerators depends on the equations of the relativistic dynamics of charged particles developed in Section 5.3. The successful design of these accelerators shows that the theory developed, in which it is assumed that the inertial mass varies with velocity, that the electric charge is constant and that particle velocities are less than the velocity of light, is satisfactory up to very high energies.

5.5. THE TRANSFORMATION OF THE MOMENTUM OF A PARTICLE AND THE FORCE ACTING ON A PARTICLE

In Sections 5.3 and 5.4 the motion of charged particles in electric and magnetic fields was discussed and it was shown that when mass, momentum and force are redefined as described in Section 5.3, then the new theory is able to predict the motion of charged particles up to extremely high energies. The equations were used in the laboratory system. It is sometimes more convenient to carry out the calculations in an inertial frame which is moving relative to the laboratory, such as, for example, the centre of mass system.

5.5.1. *Transformation of Momentum and Energy*

In the inertial frame Σ the momentum of the particle is defined by the relations

$$p_x = mu_x = \frac{m_0 u_x}{\sqrt{1 - u^2/c^2}}; \quad p_y = \frac{m_0 u_y}{\sqrt{1 - u^2/c^2}}; \quad p_z = \frac{m_0 u_z}{\sqrt{1 - u^2/c^2}}$$

and the total energy is defined by the relation

$$E = mc^2 = \frac{m_0 c^2}{\sqrt{1 - u^2/c^2}}$$

The corresponding quantities in Σ' are defined as

$$p_x' = m' u_x' = \frac{m_0 u_x'}{\sqrt{1 - u'^2/c^2}}; \quad p_y' = \frac{m_0 u_y'}{\sqrt{1 - u'^2/c^2}}; \quad p_z' = \frac{m_0 u_z'}{\sqrt{1 - u'^2/c^2}}$$

and

$$E' = m'c^2 = \frac{m_0c^2}{\sqrt{1 - u'^2/c^2}}$$

Substituting for $1/(1 - u'^2/c^2)^{\frac{1}{2}}$ from eqn (4.14) and substituting $u_x' = (u_x - v)/(1 - vu_x/c^2)$ [eqn (4.5)] into the expression for p_x', one obtains

$$p_x' = \frac{m_0(u_x - v)}{(1 - vu_x/c^2)} \frac{(1 - vu_x/c^2)}{\sqrt{1 - v^2/c^2}\sqrt{1 - u^2/c^2}}$$

$$= \frac{m_0}{\sqrt{1 - u^2/c^2}} \quad \frac{(u_x - v)}{\sqrt{1 - v^2/c^2}}$$

$$= \gamma(mu_x - mv)$$

But $$mu_x = p_x \text{ and } m = E/c^2$$

Hence $$p_x' = \gamma(p_x - vE/c^2) \tag{5.54}$$

Substituting for $1/(1 - u'^2/c^2)^{\frac{1}{2}}$ from eqn (4.14) and substituting $u_y' = u_y \dfrac{\sqrt{1 - v^2/c^2}}{(1 - vu_x/c^2)}$ into the expression for p_y', one obtains

$$p_y' = \frac{m_0}{\sqrt{1 - u'^2/c^2}} u_y' = \frac{m_0(1 - vu_x/c^2)}{\sqrt{1 - v^2/c^2}\sqrt{1 - u^2/c^2}} \frac{u_y\sqrt{1 - v^2/c^2}}{(1 - vu_x/c^2)}$$

$$= \frac{m_0u_y}{\sqrt{1 - u^2/c^2}} = p_y$$

that is

$$p_y' = p_y \tag{5.55}$$

Similarly

$$p_z' = p_z \tag{5.56}$$

Now

$$E' = m'c^2 = \frac{m_0c^2}{\sqrt{1 - u'^2/c^2}} = \frac{m_0(1 - vu_x/c^2)\, c^2}{\sqrt{1 - v^2/c^2}\sqrt{1 - u^2/c^2}}$$

$$= \gamma mc^2 \left(1 - \frac{vu_x}{c^2}\right)$$

$$E' = \gamma(E - vp_x) \tag{5.57}$$

Hence

$$\frac{E'}{c^2} = \gamma\left(\frac{E}{c^2} - \frac{vp_x}{c^2}\right) \tag{5.58}$$

Similarly one has the inverse relations

$$p_x = \gamma\left(p'_x + \frac{vE'}{c^2}\right) \tag{5.59}$$

$$p_y = p'_y \tag{5.60}$$

$$p_z = p'_z \tag{5.61}$$

$$E = \gamma(E' + vp'_x) \tag{5.62}$$

It will be seen that p'_x, p'_y, p'_z and E'/c^2 transform in the same way as the space and time co-ordinates x', y', z', t'. For example, corresponding to eqn (5.54), $x' = \gamma(x - vt)$, and corresponding to eqn (5.58), $t' = \gamma(t - vx/c^2)$

Now

$$\begin{aligned} p'^2 - E'^2/c^2 &= p'^2_x + p'^2_y + p'^2_z - E'^2/c^2 \\ &= \gamma^2(p_x - vE/c^2)^2 + p_y^2 + p_z^2 - \gamma^2(E - vp_x)^2/c^2 \\ &= p_x^2 + p_y^2 + p_z^2 - E^2/c^2 \\ &= p^2 - E^2/c^2 \end{aligned} \tag{5.63}$$

Hence, the quantity $p^2 - E^2/c^2$ is an invariant. It follows from eqn (5.22) that its numerical value is equal to $-m_0^2c^2$.

A system of two colliding particles will now be considered. Let particle 1 have constant momentum $\mathbf{p}_1$ and total energy E_1 and let particle 2 have constant momentum $\mathbf{p}_2$ and total energy E_2 relative to Σ before the collision. Let the corresponding values relative to Σ' be $\mathbf{p}'_1$, E'_1, $\mathbf{p}'_2$ and E'_2 respectively.

Let

$$E = E_1 + E_2; \quad E' = E'_1 + E'_2 \tag{5.64}$$

and

$$\mathbf{p} = \mathbf{p}_1 + \mathbf{p}_2; \quad \mathbf{p}' = \mathbf{p}'_1 + \mathbf{p}'_2 \tag{5.65}$$

Using eqn (5.54)

$$\begin{aligned} (p'_1)_x + (p'_2)_x &= \gamma[(p_1)_x - vE_1/c^2] + \gamma[(p_2)_x - vE_2/c^2] \\ &= \gamma[(p_1)_x + (p_2)_x - v(E_1 + E_2)/c^2] \end{aligned}$$

or

$$p'_x = \gamma[p_x - vE/c^2] \tag{5.66}$$

Similarly,

$$p'_y = p_y; \quad p'_z = p_z; \quad E' = \gamma[E - vp_x] \tag{5.67}$$

Hence, the sum of the momenta of the two particles and the sum of the total energies of the two particles before the collision transform in the same way as the momentum and total energy of a single particle. By repeating the argument leading to eqn (5.63), we have

for the sum of the momenta and the sum of the total energies of the two particles before collision

$$p'^2 - E'^2/c^2 = p^2 - E^2/c^2 \tag{5.68}$$

5.5.2. *The Transformation of Force*

In the inertial frame Σ the force acting on a particle is defined as a vector

$$\mathbf{f} = \mathrm{d}\mathbf{p}/\mathrm{d}t \tag{5.69}$$

having components

$$f_x = \mathrm{d}p_x/\mathrm{d}t \quad \text{etc.} \tag{5.70}$$

Similarly, in the inertial frame Σ', the force acting on the particle is defined as a vector having components

$$f_x' = \mathrm{d}p_x'/\mathrm{d}t' \quad \text{etc.} \tag{5.71}$$

Substituting for p_x' from eqn. (5.54) into eqn. (5.71)

$$f_x' = \gamma \frac{\mathrm{d}}{\mathrm{d}t'}\left(p_x - \frac{vE}{c^2}\right) = \gamma \frac{\mathrm{d}t}{\mathrm{d}t'}\frac{\mathrm{d}}{\mathrm{d}t}\left(p_x - \frac{vE}{c^2}\right) \tag{5.72}$$

Now,

$$\frac{\mathrm{d}t}{\mathrm{d}t'} = \frac{1}{\mathrm{d}t'/\mathrm{d}t} = \frac{1}{\dfrac{\mathrm{d}}{\mathrm{d}t}\gamma(t - vx/c^2)} = \frac{1}{\gamma(1 - vu_x/c^2)} \tag{5.73}$$

and

$$\mathrm{d}p_x/\mathrm{d}t = f_x$$

Using eqn (5.15)

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(T + m_0c^2) = \frac{\mathrm{d}T}{\mathrm{d}t} = \mathbf{f}\cdot\mathbf{u} = f_xu_x + f_yu_y + f_zu_z$$

Substituting in eqn (5.72) and rearranging, we obtain

$$f_x' = f_x - \frac{vu_y}{(c^2 - vu_x)}f_y - \frac{vu_z}{(c^2 - vu_x)}f_z \tag{5.74}$$

Since,

$$p_y' = p_y \tag{5.55}$$

$$f_y' = \frac{\mathrm{d}p_y'}{\mathrm{d}t'} = \frac{\mathrm{d}p_y}{\mathrm{d}t'} = \frac{\mathrm{d}t}{\mathrm{d}t'}\frac{\mathrm{d}p_y}{\mathrm{d}t} = \frac{\mathrm{d}t}{\mathrm{d}t'}f_y$$

Using eqn. (5.73), we obtain

$$f_y' = \frac{c^2\sqrt{1 - v^2/c^2}}{(c^2 - vu_x)}f_y \tag{5.75}$$

Similarly,

$$f'_z = \frac{c^2\sqrt{1 - v^2/c^2}}{(c^2 - vu_x)} f_z \tag{5.76}$$

The inverse relations are

$$f_x = f'_x + \frac{vu'_y}{(c^2 + vu'_x)} f'_y + \frac{vu'_z}{(c^2 + vu'_x)} f'_z \tag{5.77}$$

$$f_y = \frac{c^2\sqrt{1 - v^2/c^2}}{(c^2 + vu'_x)} f'_y \tag{5.78}$$

$$f_z = \frac{c^2\sqrt{1 - v^2/c^2}}{(c^2 + vu'_x)} f'_z \tag{5.79}$$

All the transformations derived in this section relate to the instantaneous values of the mass, energy and momentum of the particle and the force acting on the particle at some identifiable point on the path of the particle, the co-ordinates and time of this point in Σ and Σ' being related by the Lorentz transformations.

5.6. THE THEORY OF ELASTIC COLLISIONS

It was from a consideration of the collision of two elastic spheres that one was led to redefine the inertial mass of a moving particle to be equal to $m_0/(1 - u^2/c^2)^{\frac{1}{2}}$, where m_0 is the rest mass and u is the velocity of the particle. This led one in turn to redefine momentum and force, and using these new definitions it was possible to formulate a satisfactory theory for the relativistic dynamics of a single particle. The case of the elastic collision of two particles is now considered in more detail. By elastic collision is meant a collision in which, apart from the rest mass energies of the particles, all the energy is in the form of kinetic energy both before and after the collision. It will be *postulated* that in elastic collisions both linear momentum and kinetic energy are conserved. If it is assumed that the particles do not change their identity during the collision, then the rest masses of the colliding particles are the same before and after the collision. If the total kinetic energy is conserved in the collision then, since for each particle $E = T + m_0c^2$, the sum of the total energies of the two particles is also conserved in the collision.

Let a particle of rest mass m_0 move along the x axis with velocity $\mathbf{u}_1$, momentum $\mathbf{p}_1$ and total energy E_1 and let it collide with a particle of rest mass m_0, which is at rest in the laboratory system. The laboratory system will be referred to subsequently as the lab.-system

and denoted by Σ, as shown in *Figure 5.9(a)*. The collision will also be considered from an inertial frame Σ', which is moving with velocity v relative to Σ along the common x axis, such that the total momentum of the two particles is zero in Σ' before collision. This system is generally called the centre of mass system and denoted the c.m. system, though the term centre of momentum system

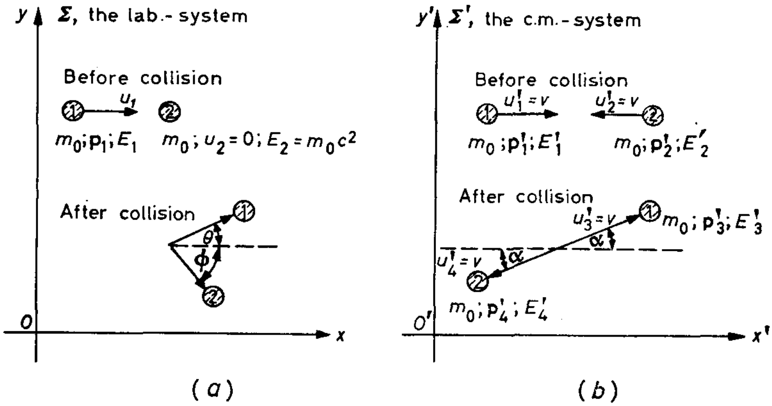


Figure 5.9. The elastic collision between two particles having equal rest masses. Particle 2 is at rest in Σ before collision, whilst particle 1 moves with velocity $\mathbf{u}_1$ relative to Σ before collision. In the c.m.-system Σ', the particles approach each other with the same velocity v before collision and rebound with equal and opposite velocities v after collision as shown in (b)

is sometimes used. In the lab.-system the total momentum before collision is given by

$$\mathbf{p} = \mathbf{p}_1 + 0 = \mathbf{p}_1$$

and the total energy before collision is given by

$$E = E_1 + E_2 = \frac{m_0c^2}{\sqrt{1 - u_1^2/c^2}} + m_0c^2 = m_0c^2\left\{\frac{1}{\sqrt{1 - u_1^2/c^2}} + 1\right\}$$

Using eqn (5.66) for the transformation of the sum of the momenta of the colliding particles from Σ to Σ' one has

$$p_x' = \gamma[p_x - vE/c^2]$$

Similarly, $p_y' = p_z' = 0$.
For the total momentum to be zero in Σ', one must have

$$p_x = vE/c^2, \text{ where } E = E_1 + E_2$$

or

$$v = \frac{p_x c^2}{E} = \frac{m_0 u_1 c^2}{\sqrt{1 - u_1^2/c^2}} \frac{1}{m_0 c^2\left\{\dfrac{1}{\sqrt{1 - u_1^2/c^2}} + 1\right\}} \tag{5.80}$$

that is

$$v = \frac{p_x c^2}{E} = \frac{u_1}{\{1 + \sqrt{1 - u_1^2/c^2}\}} \tag{5.81}$$

This is the velocity of the c.m.-system relative to the laboratory. If the x component of total momentum and the total energy in the lab.-system are conserved in the collision, then the c.m.-system must continue to move with the same velocity v relative to the lab.-system after the collision. Since, in the c.m.-system, the total momentum is zero, before collision, we have

$$\mathbf{p}_1' + \mathbf{p}_2' = 0 \quad \text{or} \quad \mathbf{p}_1' = -\mathbf{p}_2'$$

It is being assumed that the rest masses of the two colliding particles are equal so that, if their momenta are numerically equal in Σ' before collision, one must have $\mathbf{u}_1' = -\mathbf{u}_2'$, as shown in *Figure 5.9(b)*. Since particle 2 is at rest in the lab.-system (Σ) before collision, it must move along the negative $O'x'$ direction with velocity v relative to Σ' before collision as shown in *Figure 5.9(b)*. Thus in the c.m.-system the two particles (which have equal rest masses) approach each other with equal speeds v before the collision. In the c.m.-system the sum of the total energies of the particles before collision is equal to $2m_0c^2/(1 - v^2/c^2)^{\frac{1}{2}}$. If momentum is conserved in the collision in the c.m.-system, then the total momentum after the collision must be zero so that the particles must rebound with equal and opposite velocities. If energy is also conserved in the c.m.-system each particle must have a total energy equal to $m_0c^2/(1 - v^2/c^2)^{\frac{1}{2}}$ after collision, so that they must both rebound with velocity v. However, they may rebound in different directions in the c.m.-system after collision, compared with their directions before collision. Let particle 1 be scattered such that it makes an angle α with the $O'x'$ axis in the c.m.-system after collision as shown in *Figure 5.9(b)*. Choose the orientation of the y' axis such that after collision particle 1 is in the $x'y'$ plane, that is $(p_3')_z = 0$. It follows from the conservation of momentum that $(p_3')_z + (p_4')_z = p_z' = 0$ in the c.m. system, so that $(p_4')_z$ must also be zero and particle 2 must also rebound in the $x'y'$ plane in Σ'. The components of the velocity of particle 1 in Σ' after the collision are

$$(u_3')_x = v \cos \alpha; \quad (u_3')_y = v \sin \alpha; \quad (u_3')_z = 0$$

Transforming to Σ (the lab.-system) using the velocity transformations, we have

$$(u_3)_x = \frac{(u_3')_x + v}{\left(1 + \dfrac{v(u_3')_x}{c^2}\right)} = \frac{v\cos\alpha + v}{[1 + (v^2/c^2)\cos\alpha]}$$

$$(u_3)_y = \frac{(u_3')_y\sqrt{1 - v^2/c^2}}{1 + \dfrac{v(u_3')_x}{c^2}} = \frac{v\sin\alpha\sqrt{1 - v^2/c^2}}{1 + (v^2/c^2)\cos\alpha}$$

From *Figure 5.9(a)* it can be seen that

$$\tan\theta = \frac{(u_3)_y}{(u_3)_x} = \frac{\sin\alpha\sqrt{1 - v^2/c^2}}{(1 + \cos\alpha)} \tag{5.82}$$

For particle 2 in Σ', after collision,

$$(u_4')_x = -v\cos\alpha; \quad (u_4')_y = -v\sin\alpha; \quad (u_4')_z = 0$$

Using the velocity transformations,

$$(u_4)_x = \frac{-v\cos\alpha + v}{1 - (v^2/c^2)\cos\alpha}; \quad (u_4)_y = \frac{-v\sin\alpha\sqrt{1 - v^2/c^2}}{1 - (v^2/c^2)\cos\alpha}$$

$$(u_4)_z = 0$$

so that

$$\tan\phi = \frac{-(u_4)_y}{(u_4)_x} = \frac{\sin\alpha\sqrt{1 - v^2/c^2}}{(1 - \cos\alpha)} \tag{5.83}$$

Multiplying eqn (5.82) by eqn (5.83),

$$\tan\theta\tan\phi = \frac{\sin^2\alpha\,(1 - v^2/c^2)}{(1 - \cos^2\alpha)} = (1 - v^2/c^2)$$

Now from eqn (5.81)

$$v = \frac{u_1}{\{1 + \sqrt{1 - u_1^2/c^2}\}} \tag{5.84}$$

$$1 - v^2/c^2 = 1 - \frac{u_1^2}{c^2\{1 + \sqrt{1 - u_1^2/c^2}\}^2} = \frac{2}{1 + \dfrac{1}{\sqrt{1 - u_1^2/c^2}}}$$

Hence,

$$\tan\theta\tan\phi = \frac{2}{1 + \dfrac{1}{\sqrt{1 - u_1^2/c^2}}} \leq 1$$

If $u_1 \ll c$, then $\dfrac{1}{\sqrt{1 - u_1^2/c^2}} \simeq 1$ and $\tan\theta\tan\phi \simeq 1$.

6

Now

$$\tan(\theta + \phi) = \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi} \tag{5.85}$$

so that if $u_1 \ll c$, $\tan(\theta + \phi)$ becomes infinite, and

$$\theta + \phi = \frac{\pi}{2}$$

This is the expression given by Newtonian mechanics for the angle between two particles of equal rest mass after collision. It was shown by Blackett and Champion[7] that when α particles from a polonium source, for which $u_1/c \ll 1$, were scattered in a helium filled cloud chamber the angle between the α-particles after collision was equal to $\pi/2$.

If u_1 is finite then $\tan\theta \tan\phi$ is less than unity, so that in eqn (5.85) $\tan(\theta + \phi)$ is finite and $(\theta + \phi)$ is less than $\pi/2$. Champion[8] investigated the scattering of electrons, having velocities in the range $0{\cdot}82c$ to $0{\cdot}94c$, by electrons in a cloud chamber filled with either nitrogen or oxygen. The momenta of the electrons before and after collision were determined from their curvatures in a magnetic field. Champion found that $(\theta + \phi)$ was less than $\pi/2$, generally by about 10 degrees or more. The results could not be interpreted as elastic collisions between electrons on the basis of Newtonian mechanics, but the observed values for $(\theta + \phi)$ were in very good agreement with the predictions of the theory of special relativity for elastic collisions between electrons. Champion showed that the variation of $(\theta + \phi)$ from $\pi/2$ was in better agreement with the theory of special relativity than with the formula for the variation of mass with velocity suggested by Abraham (1903).

The theory of special relativity says nothing about the probability that the two electrons collide or the probability of obtaining various scattering angles, but merely gives relations between the momenta of the particles after collision. For a more complete theory of the scattering process one would have to use relativistic quantum mechanics. It must be pointed out that, in our calculations, it was assumed that one could identify particles 1 and 2 before and after collision. According to quantum mechanics this is not possible, if both the particles are electrons.

The solution of the general case of the elastic collision between two particles was given by Jüttner[9] in 1914. Let the momenta of the colliding particles be $\mathbf{p}_1$ and $\mathbf{p}_2 = 0$ in the lab.-system before collision. Let their rest masses be m_0 and M_0 respectively and let

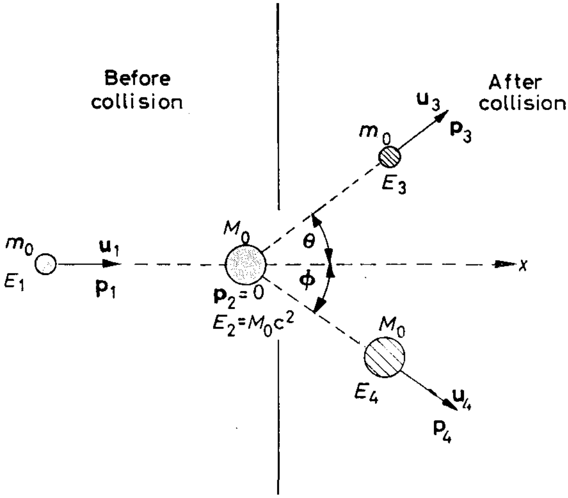


Figure 5.10. The elastic collision between a particle of rest mass m_0 moving with velocity $\mathbf{u}_1$ and a stationary particle of rest mass M_0

their total energies before collision be E_1 and E_2 respectively. Let particle 1 have momentum $\mathbf{p}_3$ after the collision and be scattered through an angle θ in the lab.-system as shown in *Figure 5.10.* Let the struck particle rebound at an angle ϕ with momentum $\mathbf{p}_4$ in the lab.-system. According to the law of conservation of momentum

$$\mathbf{p}_1 + 0 = \mathbf{p}_3 + \mathbf{p}_4$$

Writing this out into components,

$$p_1 = p_3 \cos \theta + p_4 \cos \phi \tag{5.86}$$

$$0 = p_3 \sin \theta - p_4 \sin \phi \tag{5.87}$$

The law of conservation of energy gives

$$E_1 + E_2 = E_3 + E_4$$

that is

$$c\sqrt{p_1^2 + m_0^2c^2} + M_0c^2 = c\sqrt{p_3^2 + m_0^2c^2} + c\sqrt{p_4^2 + M_0^2c^2} \tag{5.88}$$

If it is assumed that m_0, $\mathbf{p}_1$, M_0, $\mathbf{p}_2 = 0$ are given then there are four unknowns p_3, p_4, θ and ϕ in the three equations (5.86), (5.87) and (5.88). One of the four unknowns must be given if the problem is to be solved. For example, if the scattering angle θ

is given, then after some lengthy algebra one obtains:

$$p_3 = p_1 \left\{ \frac{(m_0^2c^2 + M_0E_1)\cos\theta + (E_1 + M_0c^2)\sqrt{M_0^2 - m_0^2\sin^2\theta}}{\left(\dfrac{E_1}{c} + M_0c\right)^2 - p_1^2\cos^2\theta} \right\} \tag{5.89}$$

$$E_3 = \frac{(E_1 + M_0c^2)(m_0^2c^2 + M_0E_1) + c^2p_1^2\cos\theta\sqrt{M_0^2 - m_0^2\sin^2\theta}}{\left(\dfrac{E_1}{c} + M_0c\right)^2 - p_1^2\cos^2\theta} \tag{5.90}$$

$$p_4^2 = p_1^2 + p_3^2 - 2p_1p_3\cos\theta \tag{5.91}$$

Alternatively, the energies and momenta after collision can be expressed in terms of ϕ, the angle of recoil. One then has

$$p_4 = \frac{p_1 2M_0(E_1 + M_0c^2)\cos\phi}{\left(\dfrac{E_1}{c} + M_0c\right)^2 - p_1^2\cos^2\phi} \tag{5.92}$$

$$E_4 = M_0c^2 + \frac{2p_1^2\cos^2\phi M_0c^2}{\left(\dfrac{E_1}{c} + M_0c\right)^2 - p_1^2\cos^2\phi} \tag{5.93}$$

$$p_3^2 = p_1^2 + p_4^2 - 2p_1p_4\cos\phi \tag{5.94}$$

A comprehensive discussion of eqns (5.89)–(5.94) is given by Jánossy[10].

The elastic scattering of two particles of unequal rest masses can also be discussed in the c.m.-system. The approach is similar to that adopted when discussing the example of the elastic collision of two particles of equal mass. The momenta of the colliding particles are again equal in the c.m.-system, but, since the rest masses of the particles differ, they do not approach each other with equal speeds in the c.m.-system. If the collision is elastic, the particles rebound with the same numerical values of momenta as they had before collision. The angle α the particles make in the c.m.-system with the direction of motion of the c.m.-system relative to the lab.-system is again a variable parameter. Experimental data on the elastic scattering of nuclear particles are often presented in terms of the probability of scattering between angles α and $\alpha + d\alpha$ in the c.m.-system. The momenta and energies in the c.m.-system can be related to the lab. values using the momentum and energy transformations [eqns (5.59)–(5.62)]. For low energy collisions ($<$ 20 MeV or so for nucleons, i.e. protons and neutrons) Newtonian mechanics and the Galilean transformations are sufficiently accurate.

5.7. INELASTIC COLLISIONS AND THE EQUIVALENCE OF MASS AND ENERGY

In Section 5.6 elastic collisions were considered in which, apart from the rest mass energy, all the energy before and after collision was in the form of kinetic energy. In Newtonian mechanics, kinetic energy is not always conserved in collisions, but sometimes some of the kinetic energy is converted into heat and sound, whilst in other cases some of the kinetic energy may be transformed into potential energy. The law of conservation of momentum is always

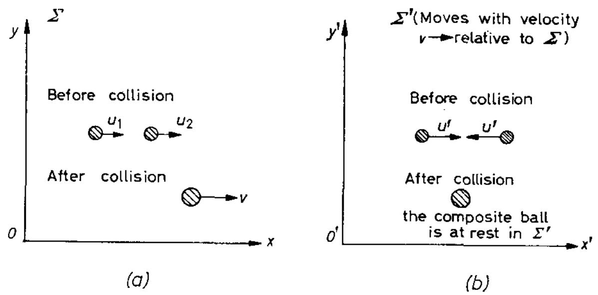


Figure 5.11. The two spheres approach each other with equal speeds u' in Σ' and are brought to rest relative to Σ' by the collision as shown in (b). The composite ball moves with uniform velocity v relative to Σ as shown in (a)

valid within the limitations of Newtonian mechanics, and the law of conservation of energy also holds provided *all* forms of energy are included. It is now seen if the laws of conservation of energy and of momentum are applicable in the general case of the collisions of high energy particles when kinetic energy is not necessarily conserved. To start with, the example of a completely inelastic collision is considered, in which, in the inertial frame Σ', there is no kinetic energy after collision.

Let two spheres, which have the same rest mass m_0, move with velocity $+u'$ and $-u'$ respectively parallel to the x' axis of the inertial frame Σ' and let them collide head-on as shown in *Figure 5.11(b)*. Let the spheres stick together and form one composite ball. Since the total momentum before collision is zero, then, if momentum is conserved in the collision, the composite ball must be at rest in Σ' after collision. The sum of the energies of the spheres before collision in Σ' is equal to

$$E' = 2m'c^2 = \frac{2m_0c^2}{\sqrt{1 - u'^2/c^2}} = 2m_0c^2 + T' \qquad (5.95)$$

where T' is the sum of the kinetic energies of the colliding particles before collision. If it is assumed that *all* the kinetic energy in Σ' goes into heat, then the heat produced in the collision, measured in Σ', is equal to

$$Q' = T' = E' - 2m_0c^2 \tag{5.96}$$

Now consider the same collision from the inertial frame Σ, where Σ moves with uniform velocity $-v$ along the x' axis relative to Σ' as shown in *Figure 5.11(a)*. The total momentum of the two spheres in Σ can be calculated from the total momentum and energy of the two spheres in Σ' just before collision, using the inverse of eqn (5.66). One has

$$p_x = \gamma(p_x' + vE'/c^2) = \frac{\gamma vE'}{c^2}$$

since the total momentum in Σ' is zero. Substituting for E' from eqn (5.96) one has, if $Q' = T'$,

$$p_x = \gamma v\left(2m_0 + \frac{T'}{c^2}\right) = \gamma v(2m_0 + Q'/c^2) \tag{5.97}$$

Similarly, from eqn (5.67)

$$p_y = p_z = 0 \tag{5.98}$$

Similarly, the total energy in Σ before collision can be calculated using the inverse of eqn (5.67). One has

$$E = \gamma(E' + vp_x') = \gamma(E' + 0) = \gamma E'$$

Substituting for E' from eqn (5.96), one obtains for the total energy in Σ *before* collision

$$E = \gamma(2m_0c^2 + T') \tag{5.99}$$

After the collision the composite ball, which is at rest in Σ', moves with velocity v in Σ. If no matter is destroyed in the collision, then one *might* expect that the rest mass of the composite ball in Σ might be equal to $2m_0$, in which case the total linear momentum in Σ after collision would be equal to

$$p_x = \frac{2m_0v}{\sqrt{1 - v^2/c^2}} = 2\gamma m_0v \tag{5.100}$$

Comparing this with eqn (5.97), it can be seen that if the rest mass were $2m_0$, then momentum would not be conserved in the collision. In order to conserve momentum in Σ, the quantity $2m_0$ in eqn (5.100) would have to be replaced by $M_0 = (2m_0 + Q'/c^2)$. Thus

a linear momentum equal to $Q'v/[c^2(1 - v^2/c^2)^{\frac{1}{2}}]$ would have to be attributed to the heat generated in the collision, if momentum is to be conserved in Σ. This heat moves with the composite ball with velocity v in Σ and its momentum would be equivalent to that of an inertial mass equal to $Q'/[c^2(1 - v^2/c^2)^{\frac{1}{2}}]$ in Σ, and an inertial mass of Q'/c^2 in Σ'. The total energy of the composite ball in Σ after collision would then be equal to $M_0c^2/(1 - v^2/c^2)^{\frac{1}{2}} = \gamma(2m_0c^2 + Q')$. Comparing with eqn (5.99) it will be seen that the total energy would also be conserved, since it is being assumed that $T' = Q'$ in Σ'.

If the spheres were perfectly elastic, they would be brought to rest momentarily in Σ' and the kinetic energy T' would be converted into elastic energy. If momentum is to be conserved *during* the collision in Σ, then an inertial mass and a momentum would have to be attributed to this elastic energy. In a perfectly elastic collision the spheres would then rebound under the influence of the elastic forces developed such that they would go back along their original paths with velocities $-u'$ and $+u'$ relative to Σ'.

It has been shown that if momentum is to be conserved in 'point' collisions, then according to the theory of special relativity inertial mass and momentum must be attributed to all the forms of energy considered, such that if the energy is ΔE, then the inertial mass associated with it is $\Delta m = \Delta E/c^2$. If this is correct for all forms of energy, then, when the energy of a system changes, the inertial mass of the system should also change. Conversely, there should be an energy change associated with a change of inertial mass. It is consistent with the theory of special relativity to extend this principle to apply to the rest masses of particles and to postulate the equivalence of mass and energy in all cases. Whether this equivalence holds in practice is, of course, a point which must be checked by experiment. Some examples of the equivalence of mass and energy will now be considered.

5.8. SOME EXAMPLES OF THE EQUIVALENCE OF MASS AND ENERGY

5.8.1. *The Binding Energy of the Nucleus**

The protons and neutrons making up the nuclei of atoms are held together by attractive forces. Work must generally be done to remove protons and neutrons from a stable nucleus, so that the potential energy of the particles inside the nucleus must be negative and this negative potential energy should have 'negative inertial mass' associated with it, that is, one would expect the inertial mass

* The masses quoted are all based on the ^{16}O scale.

of a nucleus to be less than the sum of the rest masses of the individual protons and neutrons making up the nucleus. This is confirmed by the values of the atomic weights of isotopes determined experimentally using mass spectrographs. The mass defect is defined as

$$\Delta m = Zm_H + (A - Z)m_n - M_{Z,A}$$

where $m_H = 1{\cdot}008142$ a.m.u. is the mass of a hydrogen atom, $m_n = 1{\cdot}008982$ a.m.u. is the mass of a free neutron and $M_{Z,A}$ is

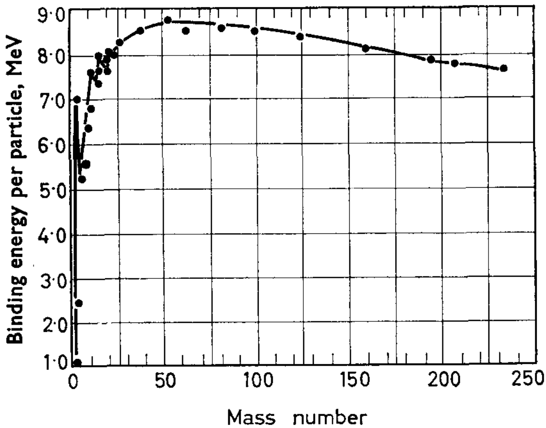


Figure 5.12. The mean binding energy per particle for the most stable nucleus at each mass number

the atomic weight of an atom containing Z protons and $A - Z$ neutrons. The mass defect is due to the binding energy of the nucleus. If the formula $\Delta E = \Delta mc^2$ is correct, then the total binding energy of the nucleus is given by

$$\Delta E(\text{MeV}) = 931[1{\cdot}008142Z + 1{\cdot}008982\,(A - Z) - M_{Z,A}]$$

using the fact that 1 a.m.u. is equivalent to 931 MeV. The average binding energy per proton or neutron can be obtained by dividing by the mass number A. The results for various nuclei are shown in *Figure 5.12.* The binding energy is negative.

The first experimental verification of the relation $\Delta E = \Delta mc^2$ was obtained by Cockroft and Walton[11] who accelerated protons up to an energy of 0·25 MeV. Cockroft and Walton observed that when these protons were incident upon a lithium target, two α-particles were sometimes emitted, the α-particles going in opposite directions. From the range of the α-particles, Cockroft and Walton

estimated that the kinetic energy of each α-particle was 8·6 MeV. The gain in kinetic energy in the reaction was equal to

$$(2 \times 8{\cdot}6 - 0{\cdot}25)\ \text{MeV} = 16{\cdot}95\ \text{MeV}$$

where the 0·25 MeV represents the kinetic energy of the incident proton. Cockroft and Walton suggested that the reaction taking place was

$${}^{7}_{3}\text{Li} + {}^{1}_{1}\text{H} \rightarrow {}^{8}_{4}\text{Be} \rightarrow {}^{4}_{2}\text{He} + {}^{4}_{2}\text{He}$$

Using the values of atomic masses available at the time Cockroft and Walton estimated that

$${}^{7}_{3}\text{Li} + {}^{1}_{1}\text{H} - 2{}^{4}_{2}\text{He} = (0{\cdot}0154 \pm 0{\cdot}003)\ \text{a.m.u.}$$

This mass difference was equivalent to an energy difference of $(14{\cdot}3 \pm 2{\cdot}7)$ MeV if the relation $\Delta E = \Delta mc^2$ were correct. This was in reasonable agreement with the observed increase of kinetic energy of 16·95 MeV. The agreement is improved if more recent values for the atomic masses are used. Many similar experiments have been performed, and in all cases it is possible to estimate the energy released or absorbed in a nuclear reaction from the difference between the atomic masses of the particles before and after the reaction, using the relation $\Delta E = \Delta mc^2$.

When protons and neutrons are brought together to form nuclei, since the binding energy is negative, energy must be released; for example, if one could 'mix' two protons with two neutrons to form a helium nucleus the loss of mass would be $2{}^{1}_{1}\text{H} + 2{}^{1}_{0}\text{n} - {}^{4}_{2}\text{He} = 0{\cdot}030$ a.m.u., and an equivalent amount of energy equal to 28 MeV would be released. This type of nuclear reaction is called a fusion or a thermonuclear reaction. It is believed that thermonuclear reactions are important in energy production in stars. It is believed that there are two main cycles of nuclear reactions, namely the proton–proton chain which predominates in small stars, less massive than the sun, and the C, N, O cycle which predominates in larger stars (Gamow and Critchfield[12]). The net result, in each case, is to produce a helium nucleus from four protons with a release of 26·7 MeV of energy per helium nucleus formed.

From the energy reaching the earth from the sun it is possible to estimate the total energy radiated by the sun. The amount radiated by the sun is equal to 4×10^{26} J/sec. Hence

$$\Delta m = \frac{\Delta E}{c^2} = \frac{4 \times 10^{26}}{9 \times 10^{16}} \sim 4{\cdot}4 \times 10^{9}\ \text{kg} \sim 4{\cdot}5 \times 10^{6}\ \text{tons}$$

Thus the mass of the sun decreases by over four million tons per second. This is small compared with the total mass of the sun which is $1{\cdot}98 \times 10^{30}$ kg or $2{\cdot}1 \times 10^{27}$ tons.

Another important type of nuclear reaction is fission. It will be seen from *Figure 5.12* that the average binding energy per nucleon (proton or neutron) in the nucleus decreases at very high values of atomic weights. The protons and neutrons are not as tightly bound in very large nuclei such as uranium as in medium sized nuclei of mass number ~ 100. If a nucleus, such as uranium, were split in two, then since the protons and neutrons would be more tightly bound in the resulting nuclei, energy should be released. A typical example of the fission of a ^{235}U nucleus is

$$^{235}_{92}\mathrm{U} + {}^{1}_{0}\mathrm{n} \rightarrow {}^{236}_{92}\mathrm{U} \rightarrow {}^{140}_{54}\mathrm{Xe} + {}^{94}_{38}\mathrm{Sr} + 2{}^{1}_{0}\mathrm{n} + \gamma + 200 \text{ MeV}$$

5.8.2. *Kinetic Energy*

It was shown in Section 5.3 that the inertial mass of a particle changed when its kinetic energy was increased. This led to the variation of inertial mass with velocity, a result which proved to be consistent with experiments (cf. Section 5.4.4).

5.8.3. *Electromagnetic Radiation (Light Quanta)*

On classical electromagnetic theory, light is considered as a form of wave motion and it is assumed that the flux of energy in the waves is given by the Poynting vector **S**, which is related to the electric intensity **E** and the magnetizing force **H** by the relation

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}$$

It is assumed that the Poynting vector represents the amount of energy crossing unit area perpendicular to the vector **S** per second. According to the ideas developed in Section 5.7 a momentum flow should be associated with this energy flow. If this were correct, then, when light is reflected at the surface of a body, since the direction of energy and momentum flow is changed, there should be a transfer of momentum from the light to the reflector. The existence of this radiation pressure was first shown by Lebedew in 1900, before the development of the theory of special relativity.

In 1900 Planck introduced the idea that radiation is emitted in the form of discrete bundles or quanta of energy. According to Planck, if the frequency of the radiation is ν, then the energy of each quantum is equal to $h\nu$, where h is called Planck's constant and is equal to $6{\cdot}625 \times 10^{-34}$ J.sec. Such a quantum is generally called a photon, though if its energy is very high it is sometimes referred to as a γ-ray. Einstein used the photon concept to interpret the photoelectric effect and suggested that if light of frequency ν is incident on the surface of a substance, individual photoelectrons are emitted from the surface when single photons are absorbed.

According to Einstein, the kinetic energy of the electrons is equal to $h\nu - \phi$, where ϕ is the work function.

The existence of individual photons travelling with the velocity of light is now an accepted experimental fact. For a particle of rest mass m_0 one has

$$m = \frac{m_0}{\sqrt{1 - u^2/c^2}}; \quad \mathbf{p} = \frac{m_0 \mathbf{u}}{\sqrt{1 - u^2/c^2}}; \quad E = \frac{m_0 c^2}{\sqrt{1 - u^2/c^2}}$$

As u tends to c, the denominator in each case tends to zero, but if the rest mass also tends to zero, then each of the above quantities can remain finite. If $m_0 \to 0$ when $u \to c$ such that $m_0/(1 - u^2/c^2)^{\frac{1}{2}}$ equals k, then

$$m = k; \quad \mathbf{p} = k\mathbf{c}; \quad E = kc^2$$

Now for a photon, E equals $h\nu$ so that k must be equal to $h\nu/c^2$. Hence, according to the theory of special relativity a photon of energy $h\nu$ has a linear momentum $h\nu/c$ and an inertial mass $h\nu/c^2$ associated with it. It is sometimes said that a photon has zero rest mass. Since photons travel with the velocity of light, it is impossible to find an inertial reference frame in which the photons are at rest, so that the term rest mass is not strictly applicable to photons.

The above theory was applied successfully by Compton to interpret what is now known as the Compton effect. Compton showed experimentally that when a monochromatic beam of x-rays is scattered by a light element such as carbon, the scattered radiation consists of two components, one having the same wavelength as the incident radiation and the other having a slightly longer wavelength. In order to account for the presence of the longer wavelength Compton suggested that the scattering process could be treated as an elastic collision between a single photon and a free electron. Let a photon of energy $h\nu$ and momentum $h\nu/c$, moving along the positive x direction, be incident upon a stationary electron of rest mass m_0 in an inertial frame Σ as shown in *Figure 5.13*. Choose the direction of the y axis such that the scattered photon and recoiling electron are in the xy plane after the collision. Let the photon have energy $h\nu'$, and move off at an angle θ to the x axis after collision and let the electron move off at an angle ϕ with velocity $\mathbf{u}$ as shown in *Figure 5.13*. According to the law of conservation of energy:

$$h\nu + m_0 c^2 = h\nu' + mc^2 \tag{5.101}$$

where $m = m_0/(1 - u^2/c^2)^{\frac{1}{2}}$ is the relativistic mass of the electron after collision. Since $c = \lambda\nu = \lambda'\nu'$, eqn (5.101) can be rewritten as

$$mc = \frac{h}{\lambda} - \frac{h}{\lambda'} + m_0 c$$

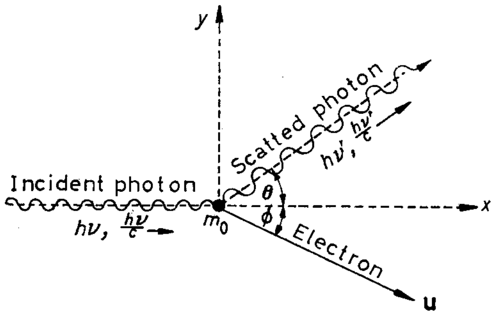


Figure 5.13. The Compton effect. A photon of energy $h\nu$ and momentum $h\nu/c$ is incident on a stationary electron of rest mass m_0. After collision a photon has energy $h\nu'$ (which is less than $h\nu$) and momentum $h\nu'/c$. The electron recoils with velocity $\mathbf{u}$

Squaring,

$$m^2c^2 = \frac{h^2}{\lambda^2} + \frac{h^2}{\lambda'^2} - \frac{2h^2}{\lambda\lambda'} + 2m_0c\left(\frac{h}{\lambda} - \frac{h}{\lambda'}\right) + m_0^2c^2$$

that is

$$(m^2 - m_0^2)c^2 = \frac{h^2}{\lambda^2} + \frac{h^2}{\lambda'^2} - \frac{2h^2}{\lambda\lambda'} + 2m_0c\left(\frac{h}{\lambda} - \frac{h}{\lambda'}\right) \tag{5.102}$$

From the law of conservation of momentum,

$$\frac{h\nu}{c} = \frac{h\nu'}{c}\cos\theta + mu\cos\phi$$

$$0 = \frac{h\nu'}{c}\sin\theta - mu\sin\phi$$

Rearranging,

$$mu\cos\phi = \frac{h}{\lambda} - \frac{h}{\lambda'}\cos\theta$$

$$mu\sin\phi = \frac{h}{\lambda'}\sin\theta$$

Squaring and adding,

$$m^2u^2 = \frac{h^2}{\lambda^2} - \frac{2h^2}{\lambda\lambda'}\cos\theta + \frac{h^2}{\lambda'^2} \tag{5.103}$$

Now,

$$m^2c^2 = m_0^2c^2/(1 - u^2/c^2)$$

that is,

$$m^2c^2 - m^2u^2 = m_0^2c^2$$

Hence $m^2u^2 = (m^2 - m_0^2)c^2$, so that the right-hand sides of eqns (5.102) and (5.103) can be equated, giving

$$-\frac{2h^2}{\lambda\lambda'} + 2m_0c\left(\frac{h}{\lambda} - \frac{h}{\lambda'}\right) = -\frac{2h^2}{\lambda\lambda'}\cos\theta$$

Hence,

$$\Delta\lambda = \lambda' - \lambda = \frac{h}{m_0c}(1 - \cos\theta) \tag{5.104}$$

Substituting numerical values for h, m_0 and c one obtains

$$\Delta\lambda = 0{\cdot}0242(1 - \cos\theta)\ \text{Å} \tag{5.105}$$

Eqn (5.105) states that when a photon is scattered through an angle θ by a free electron, the wavelength λ' of the scattered photon should be greater than the wavelength of the incident photon by an amount $0{\cdot}0242(1 - \cos\theta)$ Å. The predicted dependence of $\Delta\lambda$ on θ was verified by experiment. The kinetic energies of the recoil electrons have also been measured and found to be in agreement with the calculated values. Thus, by attributing momentum $h\nu/c$ to a photon of energy $h\nu$, Compton was able to account for the Compton effect. The calculation of the cross-section for the scattering process had to await the development of relativistic quantum mechanics.

It was shown in Section 4.4 that, using the wave theory of light, one could account for a large range of optical phenomena including the aberration of the light from stars and the Doppler effect. These phenomena will now be discussed using the photon model of light. Let a photon of energy $h\nu$ be emitted from a star at rest in an inertial frame Σ. Let the direction of the photon be towards the origin as shown in *Figure 5.14(a)*. The momentum of the photon measured in Σ has components

$$p_x = -(h\nu/c)\cos\alpha; \quad p_y = -(h\nu/c)\sin\alpha; \quad p_z = 0$$

It will be assumed that the momentum and energy of a photon transform like a single particle. In an inertial frame Σ' moving with velocity v relative to Σ as shown in *Figure 5.14(b)*, using the equation

$$p_x' = \gamma\left(p_x - \frac{vE}{c^2}\right) \tag{5.54}$$

one obtains

$$p_x' = \gamma\left(-\frac{h\nu}{c}\cos\alpha - v\frac{h\nu}{c^2}\right) = -\gamma\frac{h\nu}{c}(\cos\alpha + v/c)$$

From eqn (5.55),

$$p'_y = p_y = -\frac{h\nu}{c}\sin\alpha$$

$$\tan\alpha'_1 = \frac{p'_y}{p'_x} = \frac{\sin\alpha\sqrt{1 - v^2/c^2}}{\cos\alpha + v/c} \tag{5.106}$$

This formula is in agreement with eqns (4.24) and (4.33), which were derived in Section 4.4, using the wave model of light; eqn (5.106) represents the formula for aberration under the same conditions as considered in Section 4.4.2.

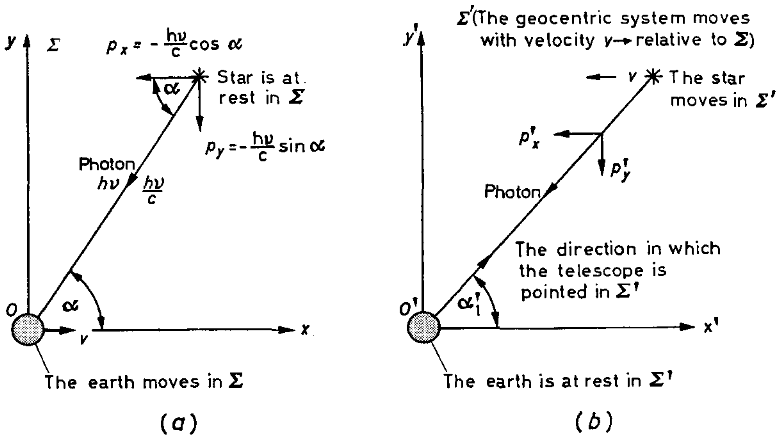


Figure 5.14. A photon is emitted from a star. (a) The photon has energy $h\nu$ and momentum $h\nu/c$ in the inertial frame Σ, in which the star is at rest. (b) The energy and momentum in the geocentric system can be calculated using the momentum and energy transformations of the theory of special relativity. This gives the correct formula for the aberration of the light from the star

In order to treat the Doppler effect, the same conditions as used in Section 4.4.4, and illustrated in *Figure 4.4* are considered. Let a source at rest at the origin of Σ' emit a photon at a time $t' = t = 0$ when the origins of Σ and Σ' coincide. Let the frequency of the photon measured in Σ' be ν', and let the frequency of the *same* photon, measured in Σ, be ν. Let the photon make an angle θ to the x axis in Σ so that its momentum has components $p_x = (h\nu/c)\cos\theta, p_y = (h\nu/c)\sin\theta$; $p_z = 0$, in Σ. From eqn (5.57),

$$E' = \gamma(E - vp_x)$$

that is,

$$h\nu' = \gamma\left(h\nu - v\frac{h\nu}{c}\cos\theta\right)$$

which reduces to

$$\nu = \frac{\nu'\sqrt{1 - v^2/c^2}}{(1 - (v/c)\cos\theta)} \tag{5.107}$$

This is in agreement with eqn (4.36). Thus the Doppler effect and the aberration of the light from stars can be accounted for satisfactorily on the photon model of light in terms of the different measures of the energies and momenta of photons in Σ and Σ'.

In addition to the photon, it is now believed that the neutrino and antineutrino also have 'zero rest mass' but unlike the photon they have spin $\frac{1}{2}(h/2\pi)$. If neutrinos have 'zero rest mass' then they must travel with the velocity of light.

5.8.4. *Pair Production and Positron Annihilation*

It would be consistent with the theory of special relativity to extend the equivalence of mass and energy to the rest masses of particles and write

$$E = m_0c^2$$

According to this relation an amount of energy equal to m_0c^2 would have to be expended in creating a particle of rest mass m_0, and if a particle of finite rest mass were annihilated then an amount of energy equal to m_0c^2 would be released. After the discovery of the positron, pair production was discovered. If a photon has energy $h\nu > 2m_0c^2$, where m_0 is the rest mass of the electron, then the photon can produce an electron pair. The process does not take place spontaneously in a vacuum but it appears that the presence of an electric charge, such as the charge on an atomic nucleus or an electron, is necessary before pair production can take place. Without the presence of the charged particle both energy and momentum could not be conserved in pair production. Experiments show that the law of conservation of linear momentum and energy do hold in pair production. Some of the radiant energy is transformed into the rest masses of the positron and the electron. If the energy of the photon exceeds $2m_0c^2$, then the excess energy is distributed between the electron and positron produced and the recoiling electric charge.

When a positron slows down it is attracted to a negative electron and forms positronium. The electron and positron move in orbits around their centre of mass in a way similar to the hydrogen atom, except that the masses of the two particles are equal in this case. In the lowest Bohr orbit the quantum numbers n and l have the values 1 and 0 respectively. If the spin of the electron and the positron are opposite in direction the singlet ^1S state is formed; if the spins are parallel then the triplet ^3S state is formed. After an

average lifetime of $\sim 10^{-10}$ sec in the ^1S state the positron and electron annihilate each other according to the reaction

$$e_+ + e_- = 2h\nu$$

Since two photons are emitted, it is possible to conserve both energy and linear momentum. The energy of each photon should be equal to $m_0c^2 = 0{\cdot}511$ MeV; this is confirmed by experiment. In the triplet ^3S state, three photons are emitted, the lifetime being $\sim 1{\cdot}5 \times 10^{-7}$ sec. Since three photons are emitted, spin and angular momentum can be conserved since the spin of the photon is one unit. If linear momentum is to be conserved, the three photons must be coplanar; this is confirmed by experiment. In this case the energy is distributed statistically between the three photons. Positron annihilation is an example of the conversion of rest-mass energy into radiant energy.

5.8.5. *Particle Decays*

When unstable fundamental particles decay at rest or in flight, the dynamics of the decay schemes can be interpreted in terms of the laws of conservation of linear momentum and conservation of mass-energy. As an example, the decay of a charged π-meson when it is at rest is considered.

When a charged π-meson decays, it generally decays into a μ-meson and one neutral particle which is now believed to be a neutrino. In the inertial frame Σ' in which the π-meson is at rest when it decays (the c.m.-system), if linear momentum is conserved, then the μ-meson and the neutrino must be emitted in opposite directions as shown in *Figure 5.15(b)*. In Σ'

$$p'_\mu = p'_\nu = p' \tag{5.108}$$

Some of the rest mass energy of the π-meson is transformed into the rest masses of the μ-meson and neutrino (if it has any rest mass), the remainder of the energy available appears as the kinetic energies of the μ-meson and the neutrino. In Σ', according to the law of conservation of energy,

$$m_\pi c^2 = E'_\mu + E'_\nu \tag{5.109}$$

or

$$m_\pi c^2 = \sqrt{\{(m_\mu c^2)^2 + p'^2_\mu c^2\}} + \sqrt{\{(m_\nu c^2)^2 + p'^2_\nu c^2\}}$$

where m_π, m_μ and m_ν are the *rest* masses of the π-meson, the μ-meson and the neutrino respectively.

From eqn (5.108),

$$c^2 p'^2_\mu = c^2 p'^2_\nu$$

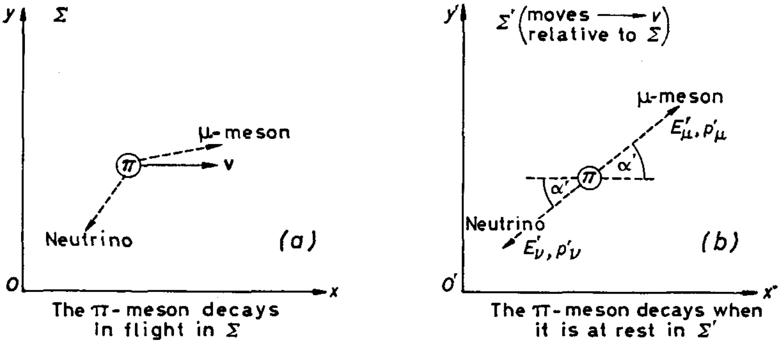


Figure 5.15. The decay of π-mesons. In Σ' the π-meson is at rest when it decays. The μ-meson and the neutrino are emitted in opposite directions in Σ' as shown in (b). The angle α' is a parameter which varies from decay to decay. In Σ the π-meson is moving with velocity v when it decays as shown in (a) and the μ-meson and neutrino are not emitted in opposite directions

so that

$$E_\mu'^2 - m_\mu^2 c^4 = E_\nu'^2 - m_\nu^2 c^4$$

or

$$(E_\mu'^2 - E_\nu'^2) = (E_\mu' + E_\nu')(E_\mu' - E_\nu') = (m_\mu^2 - m_\nu^2)c^4 \quad (5.110)$$

From eqn (5.109),

$$(E_\mu' + E_\nu') = m_\pi c^2 \quad (5.111)$$

Dividing eqn (5.110) by eqn (5.111) one obtains

$$E_\mu' - E_\nu' = \frac{(m_\mu^2 - m_\nu^2)c^2}{m_\pi} \quad (5.112)$$

Adding eqns (5.112) and (5.111), one obtains

$$E_\mu' = \frac{(m_\pi^2 + m_\mu^2 - m_\nu^2)c^2}{2m_\pi} \quad (5.113)$$

Subtracting eqn (5.112) from eqn (5.111),

$$E_\nu' = \frac{(m_\pi^2 + m_\nu^2 - m_\mu^2)}{2m_\pi}\, c^2 \quad (5.114)$$

Eqns (5.113) and (5.114) give the total energies of the μ-meson and the neutrino respectively in the inertial frame Σ' in which the π-meson is at rest when it decays. The momenta of the μ-meson and the neutrino in Σ' are given by the relation

$$cp' = \sqrt{(E_\mu'^2 - m_\mu^2 c^4)} = \sqrt{(E_\nu'^2 - m_\nu^2 c^4)} \quad (5.115)$$

If the π-meson decays in flight as shown in *Figure 5.15*(*a*), the energies and momenta of the μ-meson and neutrino in the lab.-system Σ can be calculated using the transformations for the energy and momentum of a single particle derived in Section 5.5 [eqns (5.59), (5.60), (5.61) and (5.62)]. The angle α' which the μ-meson makes with the x' axis in Σ' is a parameter which may be different in different decays. Alternatively, the laws of conservation of energy and of momentum could be applied directly in the lab.-system Σ, in which the π-meson decays in flight. As the energy of the parent π-meson is increased relative to the laboratory, the μ-mesons are emitted more and more in the direction of motion of the parent π-meson in the lab.-system.

The above analysis holds whenever an unstable particle decays into two particles. If M is the *rest* mass of the decaying particle and m_1 and m_2 are the *rest* masses of the secondary particles, then eqns (5.113), (5.114) and (5.115) become

$$E_1' = (M^2 + m_1^2 - m_2^2)c^2/2M \tag{5.116}$$

$$E_2' = (M^2 + m_2^2 - m_1^2)c^2/2M \tag{5.117}$$

$$cp' = \sqrt{(E_1'^2 - m_1^2c^4)} = \sqrt{(E_2'^2 - m_2^2c^4)} \tag{5.118}$$

Some fundamental particles decay into three particles, for example a μ-meson decays into an electron and two neutrinos. Sometimes K-mesons decay into three particles, for example the decay scheme of what was originally called a τ-meson is

$$K^+ \text{ (or } \tau^+) \rightarrow \pi^+ + \pi^+ + \pi^-$$

When the τ^+ meson decays in a nuclear emulsion, the tracks of the three π-mesons are visible, and if they end in the nuclear emulsion their energies can be estimated from their ranges in the emulsion. It is found that the three π-mesons are coplanar, as they should be if linear momentum is conserved. It is also found that, though the momentum and energy available is distributed statistically between the three π-mesons, both energy and linear momentum are conserved in the decays. It is possible to interpret the dynamics of all particle decays in terms of the laws of conservation of energy and of linear momentum. Other laws, such as the laws of conservation of charge, conservation of spin and angular momentum, conservation of baryons and conservation of leptons appear to be valid also. In three-particle decays energy and momentum are distributed statistically between the decay particles. Whilst these energies and momenta are consistent with the laws of conservation of linear momentum and conservation of energy, these laws alone say nothing

about the probabilities of various energy values. For example, in order to estimate the energy distribution of electrons from μ-meson decay, one must use a precise meson theory.

Example: A π-meson of rest mass 139·6 MeV/c^2 decays at rest into a μ-meson of rest mass 105·7 MeV/c^2 and a neutrino of zero rest mass. Calculate the kinetic energy of the μ-meson. If the π-meson was moving with a velocity of $0{\cdot}8c$ relative to the laboratory when it decayed, calculate the maximum and minimum possible values for the kinetic energy of the μ-meson relative to the laboratory.

Substituting for $m_\pi c^2$, $m_\mu c^2$ and $m_\nu c^2$ into eqn (5.113),

$$E'_\mu = \frac{139{\cdot}6^2 + 105{\cdot}7^2 - 0^2}{2 \times 139{\cdot}6} = 109{\cdot}8 \text{ MeV}$$

$$T'_\mu = 109{\cdot}8 - m_\mu c^2 = 4{\cdot}1 \text{ MeV}$$

Similarly for the neutrino,

$$E'_\nu = T'_\nu = 29{\cdot}8 \text{ MeV}$$

Now,

$$cp = \sqrt{E'^2_\mu - m_\mu^2 c^4} = \sqrt{109{\cdot}8^2 - 105{\cdot}7^2} = 29{\cdot}8 \text{ MeV}$$

If the π-meson is moving when it decays, the maximum and minimum values for the kinetic energy of the μ-meson are when the μ-meson is emitted in the forward and the backward directions respectively in the c.m.-system, that is parallel and antiparallel to the x' axis in *Figure 5.15(b)*.
Since

$$u/c = 0{\cdot}8, \qquad \gamma = \tfrac{5}{3}$$

$$(E_\mu)_{\max} = \gamma(E'_\mu + vp'_x) = \tfrac{5}{3}(109{\cdot}8 + 0{\cdot}8 \times 29{\cdot}8)$$

$$(E_\mu)_{\max} = 223 \text{ MeV}; \qquad (T_\mu)_{\max} = 117 \text{ MeV}$$

$$(E_\mu)_{\min} = \tfrac{5}{3}(109{\cdot}8 - 0{\cdot}8 \times 29{\cdot}8) = 143 \text{ MeV}$$

$$(T_\mu)_{\min} = 37 \text{ MeV}$$

In the general case when α' is not 0 or 180 degrees in *Figure 5.15(b)*, the energy, momentum and direction of the μ-meson in Σ can also be calculated using the energy and momentum transformations.

5.8.6. *Experimental Check of the Principle of the Constancy of the Speed of Light in Empty Space*

In high energy nuclear interactions, π°-mesons are sometimes produced. These π°-mesons have a rest mass of 135 MeV/c^2. They are unstable and generally decay into two γ-rays (photons). In the rest frame of the decaying π°-mesons, the γ-rays have energies of

67·5 MeV each. The mean lifetime of π°-mesons, when they are at rest, is $\sim 0{\cdot}7 \times 10^{-16}$ sec. In the rest frame of the decaying π°-mesons, the speed of the photons arising from π°-meson decay should be equal to the accepted value for the speed of light emitted by a stationary source, namely $c = 2{\cdot}9979 \times 10^8$ m/sec. If the π°-mesons are moving relative to the laboratory when they decay, according to the principle of the constancy of the speed of light in empty space, the speed of the γ-rays should still be equal to c relative to the laboratory whatever the speed of the source of the γ-rays, (π°-mesons in this case). Alväger, Farley, Kjellman and Wallin[13] (1964) measured the velocity of γ-rays from π°-mesons of energy >6 GeV, corresponding to $(1 - u^2/c^2)^{-\frac{1}{2}} > 45$. The velocity of the photons was measured by timing the γ-rays over a known distance.

The experiment was performed with the CERN Proton Synchrotron, using protons of momenta 19·2 GeV/c. The circulating proton beam consisted of bunches of protons a few nano-sec (10^{-9} sec) long. The π°-mesons were produced in a beryllium target in the nuclear disintegrations produced by the 19·2 GeV/c protons. The γ-rays arising from π°-meson decays were observed at an angle of ~ 6 degrees to the proton direction. Bending magnets were used to sweep away charged particles. The π°-mesons decayed quickly (<1 cm) into γ-rays, so that each bunch of protons gave a bunch of γ-rays arising from π°-meson decay. The γ-rays were detected by means of a 4 mm lead absorber (which converted some of the γ-rays in each bunch into electron pairs), followed by a small plastic scintillator in coincidence with a lead-glass Cerenkov counter to detect the electron pairs from γ-rays of energy $\geqslant 6$ GeV. Two such detectors were used at a distance of $(31{\cdot}450 \pm 0{\cdot}0015)$ m apart. Some of the γ-rays in each bunch were converted into electron pairs in the first detector and some of the γ-rays in the *same* bunch were converted into electron pairs in the second detector. The time interval between the two pulses was measured electrically. The time of flight ($\sim 10^{-7}$ sec) between the two detectors was substantially longer than the length of each pulse of γ-rays. From the time of flight, the velocity of the γ-rays was determined. The result for the velocity of γ-rays of energy $\geqslant 6$ GeV from a source moving with a velocity of $0{\cdot}99975c$ (calculated using the equations of the theory of special relativity) was $(2{\cdot}9977 \pm 0{\cdot}0004) \times 10^8$ m/sec. This was the same, within experimental error, as the accepted value of $2{\cdot}9979 \times 10^8$ m/sec for the velocity of light from a stationary source. A similar experiment using quanta arising from positron annihilation was carried out by

Sadeh[14] (1963). This experiment also confirmed the principle of the constancy of the speed of light.

It was shown in Section 5.4.2 that accelerated electrons tend to a limiting speed, namely the speed of light. If all inertial frames are equivalent, and if the units of length and time are the same in all of them, then the limiting speed should be the same in all inertial frames. Otherwise, it would be possible to differentiate between the various inertial frames in terms of the different values for the limiting speed. This result was called the principle of the constancy of the limiting speed for particles. Photons, or light quanta such as γ-rays, can be considered as particles of 'zero rest mass' travelling at the limiting speed (cf. Section 5.8.3). According to the principle of the constancy of the limiting speed of particles, this limiting speed is the same and equal to c in all inertial frames (cf. Section 5.4.2). This, in the case of photons, becomes the principle of the constancy of the speed of light.

5.8.7. *Meson Production*

When fast protons or neutrons collide with atomic nuclei, π-mesons can be produced, provided the energy of the incident particle is high enough. The charged π-mesons produced have masses of 273·2 electron masses and the neutral π-mesons have masses of 264·4 electron masses. In these collisions some of the kinetic energy of the incident particle is converted into the rest mass energies of the π-mesons. At higher energies K-mesons, nucleons, negative protons, antineutrons and hyperons may be produced. Mesons can also be produced by high energy photons. Experiments show that the laws of conservation of linear momentum and energy are valid in these collisions. A reader interested in a full discussion of the dynamics and interpretation of such collisions is referred to Blaton[15], Dedrick[16] and Powell, Fowler and Perkins[17]. We shall confine our remarks to a brief discussion of some of the more important points.

It is not possible to transform all the kinetic energy of the colliding particles into some other form of energy (except when the lab.-system is the c.m.-system, for example, when two particles of equal rest masses approach each other with equal speeds). If the total momentum is not zero in the laboratory system before the collision, then, if momentum is conserved in the collision, there must still be the same total momentum present relative to the laboratory after the collision, so that some, at least, of the particles present after the collision must be moving and have kinetic energy relative to the laboratory. However, in the c.m.-system the total momentum

before the collision is zero, so that it is possible, in principle, for all the particles to be at rest relative to the c.m.-system after collision and still conserve momentum. In this case, all the kinetic energy in the c.m.-system would have been converted into some other form of energy by the collision, for example into the rest mass energy of a created particle, or into heat.

The collision of two particles will be considered. Let an incident particle of rest mass m_1, momentum p_1 and total energy E_1 collide with a stationary particle of rest mass m_2. It was shown in Section 5.5 that the sum of the momenta of the two particles before collision transforms in the same way as the momentum of a single particle. Using eqn (5.66), if p_1 is parallel to the x axis,

$$p'_x = \gamma(p - vE/c^2)$$

where, before collision $p = p_1$ and $E = (m_1^2c^4 + c^2p_1^2)^{\frac{1}{2}} + m_2c^2$. In the c.m.-system $p' = 0$, hence

$$v = u_{\text{c.m.}} = pc^2/E = \frac{p_1c^2}{(E_1 + m_2c^2)} = \frac{p_1c^2}{\sqrt{(p_1^2c^2 + m_1^2c^4)} + m_2c^2} \qquad (5.119)$$

If the momentum p and energy E are conserved in the collision in all inertial reference frames, the c.m.-system must continue to move with velocity $u_{\text{c.m.}}$ relative to the lab.-system after the collision.

It is convenient to discuss the inelastic collisions of atomic particles in terms of the energy available in the c.m.-system. From eqn (5.68), we have for the sum of the momenta and the sum of the total energies of the colliding particles in Σ and Σ' before collision

$$p^2 - E^2/c^2 = p'^2 - \frac{E'^2}{c^2} \qquad (5.68)$$

Now if the struck particle is at rest in the lab.-system before collision and the moving particle has total energy E_1, one has $E = E_1 + m_2c^2$. If Σ' is the c.m.-system, then $p' = 0$. Hence eqn (5.68) becomes

$$E'^2 = (E_1 + m_2c^2)^2 - p^2c^2 \qquad (5.120)$$

For particle 1 before collision

$$p_1^2c^2 = E_1^2 - m_1^2c^4$$

The momentum of particle 2 is zero before the collision. Substituting in eqn (5.120), and taking the square root,

$$E' = (2m_2c^2E_1 + m_1^2c^4 + m_2^2c^4)^{\frac{1}{2}} \qquad (5.121)$$

This is the total energy available in the c.m.-system. It includes the rest mass energies of the colliding particles. In the extreme

relativistic case $u_1 \simeq c$, $m_1c^2 \ll E_1$ and $m_2c^2 \ll E_1$, then the total energy E' available in the c.m.-system is $\simeq(2m_2c^2E_1)^{\frac{1}{2}}$. Thus at extremely high energies, the energy available in the c.m.-system increases approximately as the square root of E_1, the energy of the incident particle. For a proton of energy 30 GeV incident on a stationary nucleon (i.e. proton or neutron) there is about 7 GeV total energy available in the c.m.-system. For a 300 GeV proton the energy available in the c.m.-system is $\simeq$22 GeV.

Some of the total energy available in the c.m.-system goes into the rest masses of the particles present after the collision. These particles may be the same as or different from the colliding particles.

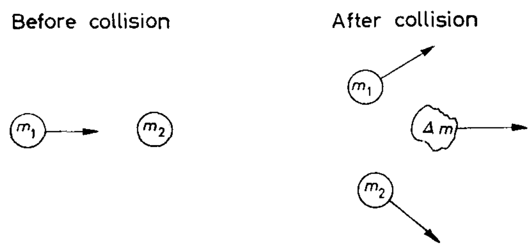


Figure 5.16. Two particles of rest masses m_1 and m_2 collide. The increase in rest mass after the collision is equal to Δm

Sometimes new particles are produced. For example, let the reaction be as shown in *Figure 5.16*. Let Δm be the sum of the rest masses of the new particles. To find the minimum energy necessary to produce the new particles, it will be assumed that all the particles are at rest in the c.m.-system after the collision. From eqn (5.121) one has for E' the total energy available in the c.m.-system

$$E'^2 = 2m_2c^2E_1 + m_1^2c^4 + m_2^2c^4 \tag{5.122}$$

After collision, if all the kinetic energy is zero in the c.m.-system, then

$$E'^2 = \{m_1c^2 + m_2c^2 + (\Delta m)c^2\}^2 \tag{5.123}$$

Equating the right-hand sides of eqns (5.122) and (5.123), one finds for $T_1 = E_1 - m_1c^2$

$$T_1 = (\Delta m)c^2\left\{1 + \frac{m_1}{m_2} + \frac{\Delta m}{2m_2}\right\} \tag{5.124}$$

This is the kinetic energy necessary in the lab.-system to produce extra particles of total rest mass Δm, if the rest masses of the colliding particles are unchanged. As an example consider the reaction

$$\gamma + \mathrm{p} \rightarrow \mathrm{p} + \pi^0$$

in which an incident photon produces a π°-meson in a collision with a stationary proton. Using p $= 1836m_e$, $\pi^{\circ} = 264{\cdot}4m_e$, one finds T_1 is equal to 145 MeV. Thus the energy of the incident photon must exceed 145 MeV if a π°-meson is to be produced in a collision with a proton.

As another example consider the reaction

$$\mathrm{p} + \mathrm{p} \rightarrow \mathrm{p} + \mathrm{n} + \pi^{+}$$

Neglecting the difference between the neutron and proton rest masses, and taking the rest mass of the π-meson as $273{\cdot}2m_e$, we find $T_1 = 290$ MeV. Thus the incident proton must have a kinetic energy exceeding 290 MeV to produce a π^{+}-meson in a proton–proton collision. It is interesting to note that the photon threshold energy is substantially less than the proton threshold energy for π-meson production. The production of a proton–antiproton pair in a proton–proton collision according to the reaction

$$\mathrm{p} + \mathrm{p} = \mathrm{p} + \mathrm{p} + \mathrm{p} + \bar{\mathrm{p}}$$

requires a proton kinetic energy of at least 5·6 GeV in the lab.-system. If the incident kinetic energy exceeds the threshold kinetic energy, the excess energy is distributed among the particles present after the collision.

5.9. TURNING LIGHT INTO GAMMA RAYS

It is sometimes convenient to consider phenomena from a different reference frame from the laboratory. In this new reference frame, the experimental conditions are different and may correspond to well known phenomena which have been studied extensively in the laboratory system. As an example of this approach, we shall consider an experiment carried out by Kulikov and colleagues[18]. In this experiment, red light of wavelength 6943 Å from a ruby laser collided head on with 500 MeV electrons in an electron synchrotron as shown in *Figure 5.17*(*a*). Using $E = h\nu$, where h is Planck's constant and ν is the frequency of the light, it follows that the energies of the incident photons are *about* 2 eV in the lab.-system.

Now consider the collision from the reference frame Σ' shown in *Figure 5.17*(*b*), in which the electrons are at rest before the collision. This reference frame moves with velocity v relative to Σ, where v is the speed of the electrons relative to the laboratory. From eqn (5.57)

$$E' = \gamma(E - vp_x) \tag{5.125}$$

where E is the energy and $\mathbf{p}$ is the momentum of the photon relative to the laboratory and $\gamma = (1 - v^2/c^2)^{-\frac{1}{2}}$. For the incident photon, let $E = h\nu = 2$ eV; $p_x = -h\nu/c$; p_x is negative since the photon moves in the negative x direction in *Figure 5.17(a)*. Substituting in eqn (5.125)

$$E' = \gamma\left(h\nu + \frac{vh\nu}{c}\right) = 2\gamma\left(1 + \frac{v}{c}\right) \text{ eV} \tag{5.126}$$

Now the energy of the electron relative to the laboratory is $\gamma m_0c^2 \sim 500$ MeV. Since for an electron, $m_0c^2 \sim 0{\cdot}5$ MeV, γ is

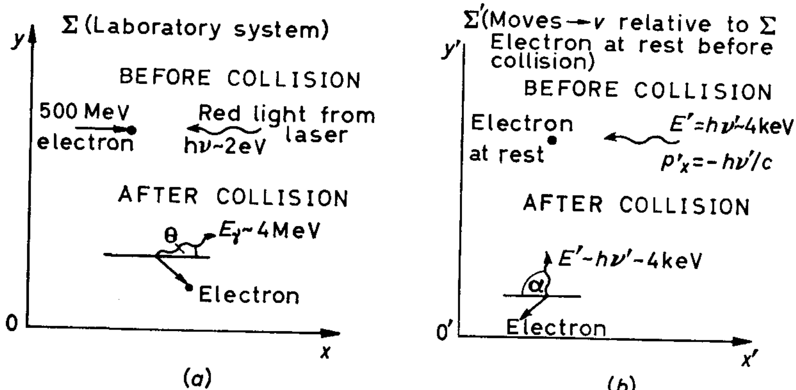


Figure 5.17. (a) Red light from a ruby laser collides with 500 MeV *electrons. The photons present after collision have energy in the* MeV *range. (b) The electrons are at rest in Σ' before collision. The photons from the laser have energies in the* keV *range in Σ'. The normal formulae for the Compton effect are valid in Σ'*

$\sim$1,000 and $v \simeq c$. Substituting in eqn (5.126), we find $E' \sim 4{,}000$ eV or $\sim$4 keV. Thus in the reference frame in which the electrons are at rest before the collision, the incident photons from the laser have energies $\sim$4 keV, which is a typical value for X-rays. Since the electron is at rest in Σ', the normal formulae for the Compton effect should hold. Applying conservation of energy and linear momentum, it was shown in Section 5.8.3 that

$$\Delta\lambda' = \frac{h}{m_0c}(1 - \cos\alpha) = 0{\cdot}0242(1 - \cos\alpha) \text{ Å}$$

where $\Delta\lambda'$ is the change in wavelength, and α is the angle of scattering in Σ'. For example, if $\alpha = \pi/2$, in Σ', $\Delta\lambda'$ is $\sim$0·0242 Å. This is small compared with the wavelength of a 4 keV photon, which is $\sim$3·1 Å. Hence the changes in wavelength and frequency are small, so that the change in the energy of the photon is small.

The reader can verify this by putting $\alpha = \pi/2$ in eqn (5.90) and showing that the energy of the photon present after a collision with a stationary electron of rest mass m_0 is $E'/(1 + E'/m_0c^2) = E'/(1 + 4/500)$. Thus in Σ', we have photons of energy $\sim$4 keV scattered in various directions after collision with stationary electrons, in accordance with the theory of the Compton effect. As a typical example take $\alpha = 90$ degrees. For the photon present after collision, in Σ' we then have, $E' = 4$ keV, $p'_x = 0$, $p'_y = E'/c$. Applying the inverse transformation to transform back to the lab.-system, we have

$$E = \gamma(E' + vp'_x) = 1{,}000E' \simeq 4 \text{ MeV}$$

Applying the momentum transformations

$$p_y = p'_y = E'/c$$

$$p_x = \gamma(p'_x + vE'/c^2) = \gamma vE'/c^2 \simeq \gamma E'/c$$

Hence in *Figure 5.17*(*a*), after collision

$$\tan\theta \simeq \theta = p_y/p_x = 1/\gamma = 10^{-3} \text{ radians}$$

Similar calculations can be applied for other values of scattering angle α in the rest frame of the incident electrons. Thus relative to the laboratory, the scattered photons have energies in the MeV (or γ-ray) region. They are scattered into a narrow beam, opposite in direction to the direction of the incident light but in the direction of the incident electrons. Thus the energetics of the process discovered by Kulikov and colleagues[18] can be illustrated in terms of the Compton effect using the energy and momentum transformations of the theory of relativity.

It is interesting to note that this process may occasionally give rise to X-rays and γ-rays in stars. Fast electrons may sometimes be produced, e.g. in stars such as the Crab Nebula, and possibly in sunspots. If sufficiently fast electrons collide with visible radiation, X-rays can be produced by the above process.

5.10. ROCKET MOTION

A rocket is initially at rest relative to the inertial reference frame Σ. It is assumed that there are no external forces acting on the rocket. The rocket propels itself rectilinearly by emitting gases in the backward direction. Let the gases be emitted at a constant speed w relative to the rocket. Consider the instant when the rocket is moving with a velocity v parallel to the x axis of Σ as shown in

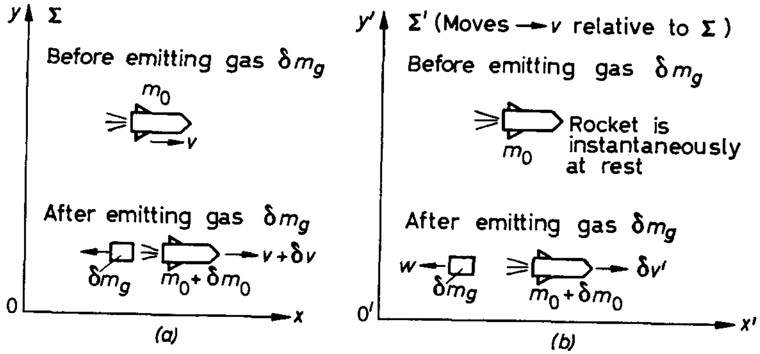


Figure 5.18. (a) The rocket is moving with velocity v relative to Σ. (b) The rocket is instantaneously at rest relative to Σ'. After emitting gases of rest mass δm_g at a speed w relative to Σ', the rocket has a speed δv' relative to Σ', its rest mass changes to $m_0 + \delta m_0$ The speed of the rocket, relative to Σ, changes to $v + \delta v$

Figure 5.18(a). At this instant, the rocket is instantaneously at rest in the inertial frame Σ', shown in *Figure 5.18(b)*. Let m_0 be the rest mass of the rocket at this instant. Let the increase in the speed of the rocket relative to Σ' be $\delta v'$ when gases of rest mass δm_g are emitted with a speed w relative to Σ'. Let the rest mass of the rocket change from m_0 to $m_0 + \delta m_0$. Application of the law of conservation of mass-energy relative to Σ' gives

$$m_0c^2 = \frac{(m_0 + \delta m_0)c^2}{\sqrt{1 - (\delta v')^2/c^2}} + \frac{\delta m_g c^2}{\sqrt{1 - w^2/c^2}}$$

Neglecting terms of order $(\delta v')^2$ gives

$$-\delta m_0 c^2 = \frac{\delta m_g c^2}{\sqrt{1 - w^2/c^2}} \tag{5.127}$$

The decrease in the rest mass-energy of the rocket $\delta m_0 c^2$ is not equal to $\delta m_g c^2$, the rest mass-energy of the emitted gases, since the gases have a velocity w and kinetic energy relative to Σ'.

Application of the law of conservation of linear momentum relative to Σ' gives

$$0 = -\frac{\delta m_g w}{\sqrt{1 - w^2/c^2}} + \frac{(m_0 + \delta m_0)\,\delta v'}{\sqrt{1 - (\delta v')^2/c^2}}$$

Neglecting terms of order $\delta m_0 \delta v'$ and $(\delta v')^2$, we have

$$\frac{\delta m_g w}{\sqrt{1 - w^2/c^2}} = m_0 \delta v' \tag{5.128}$$

Substituting for δm_g in eqn (5.127) gives

$$-\delta m_0 w = m_0\, \delta v' \tag{5.129}$$

Using the velocity transformations, eqn (4.9), since relative to Σ' the speed of the rocket is $u_x' = \delta v'$, we obtain for the velocity of the rocket relative to Σ,

$$(v + \delta v) = \frac{(\delta v' + v)}{(1 + v\,\delta v'/c^2)} \simeq (v + \delta v')\left(1 - v\frac{\delta v'}{c^2}\right)$$

$$\delta v \simeq \delta v'(1 - v^2/c^2)$$

Substituting for $\delta v'$ into eqn (5.129) gives

$$-\delta m_0 w = \frac{m_0\, \delta v}{(1 - v^2/c^2)}$$

Rearranging,

$$-\frac{w\,\delta m_0}{m_0} = \frac{\delta v}{(1 - v^2/c^2)} \tag{5.130}$$

Alternatively, the reader can apply conservation of mass-energy and conservation of momentum in the laboratory system. (Reference: Halfman[19].) Integrating,

$$-w[\ln m_0]_{m_i}^{m_f} = c\left[\frac{1}{2}\ln\left(\frac{1 + v/c}{1 - v/c}\right)\right]_0^u$$

where m_i and m_f are the initial rest mass and the final rest mass of the rocket respectively, and u is the final velocity of the rocket relative to Σ. Taking exponentials

$$\frac{m_i}{m_f} = \left(\frac{c + u}{c - u}\right)^{c/2w}$$

With present day rockets, w the exhaust speed of the gases is of the order of 10 km/sec so that $c/2w = 3 \times 10^8/2 \times 10^4 = 1{\cdot}5 \times 10^4$. If the final speed of the rocket is to be say $c/2$, then

$$\frac{m_i}{m_f} = \left(\frac{c + c/2}{c - c/2}\right)^{1\cdot5\times10^4} = (3)^{1\cdot5\times10^4} \simeq 10^{7{,}157}$$

If the final mass of the rocket is to be large enough to hold an astronaut, say 1 ton, one would have to start with a rocket weighing about $10^{7{,}157}$ tons. It can be seen that, with present day rocketry techniques, it would be impossible to accelerate a rocket to a speed comparable with the speed of light. One can accelerate atomic particles to such speeds due to their small rest mass. In these cases

the energy is applied by external methods using electric and magnetic fields. The discussion of what may or may not happen to astronauts travelling with speeds comparable to the speed of light is largely of academic interest with present day gas rockets. (If, however, one could use photons ($w = c$) or ions with ($w \simeq c$) then for $u = c/2$, m_i/m_f would only be $\sqrt{3}$.)

The acceleration of the rocket and the distance it will travel in a given time relative to Σ depend on the rate at which the exhaust gases are emitted. One would have the most comfortable journey, if the rate of emission of gases were controlled such that the acceleration of the rocket relative to the inertial frame, in which it is instantaneously at rest, were always equal to g, the terrestrial value of the acceleration due to gravity ($g \sim 9{\cdot}81$ m/sec²). It will be shown in Chapter 9 that the conditions inside the rocket would then be similar to conditions on the earth. Proceeding as in Problem 4.10, we would obtain

$$u = gt\Big/\left\{1 + \frac{g^2t^2}{c^2}\right\}^{\frac{1}{2}}$$

and

$$x = \frac{c^2}{g}\left[\left\{1 + \frac{g^2t^2}{c^2}\right\}^{\frac{1}{2}} - 1\right]$$

for the velocity and the distance travelled relative to Σ after a time t. This is an example of hyperbolic motion. In one year the rocket would travel about 0·43 light years and have a final speed of about $0{\cdot}75c$ relative to the earth. The time for the journey, relative to the rocket, can be calculated using the clock hypothesis introduced in Chapter 8 (cf. eqn (8.8) and Problems (8.8) and (8.9)).

5.11. A REVIEW OF RELATIVISTIC MECHANICS

The theory developed in this Chapter was *not* deduced from the principle of relativity and the principle of the constancy of the velocity of light. The approach adopted was to try to modify the laws of Newtonian mechanics, such that they obeyed the principle of relativity when the co-ordinates, times and velocities were transformed according to the transformations of the theory of special relativity. This necessitated a redefinition of mass, such that the mass of a body depended on its velocity, and this in turn necessitated a redefinition of momentum, force, energy, etc. The resulting theory proved to be in excellent agreement with the experimental results. The dynamics of a single particle was based

on the equation

$$\mathbf{f} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{m_0\mathbf{u}}{\sqrt{1 - u^2/c^2}}\right) = q\mathbf{E} + q\mathbf{u} \times \mathbf{B} \qquad (5.131)$$

It would be more convenient for somebody interested only in electron optics to postulate eqn (5.131) directly, rather than to start from the principle of relativity and the principle of the constancy of the velocity of light. Such an approach avoids a discussion of length contraction, time dilation, etc., since eqn (5.131) can be applied directly in the laboratory system and it is not necessary to discuss the transformation of co-ordinates and time from one reference frame to another. Such an approach has much to recommend it in an introductory course on atomic physics. The confirmation of the postulate [eqn (5.131)] would be, that predictions based on eqn (5.131) are in excellent agreement with the experimental results (cf. Appendix 4).

The consideration of collisions led to the idea of the equivalence of inertial mass and energy. In Section 5.8 a survey was given of various processes in which energy was transformed from one form to another. It was shown that changes in inertial mass were associated with changes in all forms of energy. These results can be summarized as the law of the equivalence of inertial mass and energy. Energy and inertial mass are generally interpreted as different properties of matter, for example, the energy of a particle is generally interpreted as a measure of its capacity to do work, whereas the inertial mass of a particle is a measure of the resistance of the particle to changes in its motion. The law of the equivalence of mass and energy does not imply that mass is sometimes converted into energy, or *vice versa,* but states that the changes in one are accompanied by corresponding changes in the other, inertial mass and energy being proportional to each other. The law of conservation of mass is therefore equivalent to the law of conservation of energy and they are sometimes combined together and called the law of conservation of mass-energy.

In Section 5.6 it was postulated that both linear momentum and total energy were conserved in 'point' events. It was illustrated how these conservation laws led to predictions which were in agreement with the experimental results. It was possible to calculate relationships between the momenta and energies of the particles in a collision or decay process. It should be remembered that the laws of conservation of momentum and energy do not give a complete theory of these processes. As an example, consider the Compton effect which was discussed in Section 5.8.3. The theory developed gave

the variation of frequency (and energy) of the scattered photon with angle, but did not give the probability that a collision took place, neither did it give the relative probabilities of various scattering angles. In order to obtain a fuller theory extra hypotheses would have to be introduced, and the full theory of the Compton effect had to await the development of relativistic quantum mechanics and its application to the Compton effect by Klein and Nishina[20]. However, since the new theory is consistent with the theory of special relativity, the dynamical relationships between the particles in a collision calculated using the new theory must be the same as that calculated using the techniques developed in Sections 5.7 and 5.8, subject to the limitations of the uncertainty principle.

Total energy and linear momentum were conserved in all the phenomena described in Sections 5.7 and 5.8. It appears that phenomena, such as pair production, do not take place unless there is a charged particle present so that energy and momentum can be conserved. It should be emphasized that all the phenomena considered were events taking place at approximately one point of space at one instant of time. When two particles such as two electric charges, separated in space at points x_1 and x_2 in an inertial frame Σ, give rise to electromagnetic forces on each other which lead to changes in their momenta, if momentum is conserved in one inertial frame Σ, then, if the momenta of the individual charges are transformed to Σ', the transformed values of momenta refer to times $t_1' = \gamma(t - vx_1/c^2)$ and $t_2' = \gamma(t - vx_2/c^2)$ respectively, and so do not refer to the same time in Σ'. Since the momenta are changing continuously it is not possible to formulate a law of conservation of momentum for spatially separated interacting charged particles which is valid in all inertial frames, unless one attributes momentum to the electromagnetic field. The electromagnetic forces between separated moving charges is discussed in detail by Rosser[3], Section 7.5. Similar arguments can be applied to colliding particles of finite extension. There are, however, a very wide range of phenomena which can be considered, to a good approximation, as events taking place at one point of space at one instant of time and experiments have shown that in these events the law of conservation of linear momentum and the law of conservation of mass-energy are valid. It must be remembered that the last statement must be qualified to the extent of the uncertainty principle.

After discussing the mechanics of 'point' particles, it is normal in Newtonian mechanics to extend the theory to rigid bodies of finite dimensions, the mechanics of continuous medii, hydrodynamics, etc. As fluids and large bodies almost invariably move with velocities

very much smaller than the velocity of light, the deviations from Newtonian mechanics are not important in practice. Furthermore, it is more convenient to develop these topics using tensor methods. For these reasons these topics will not be considered here, but the interested reader is referred to the books by Møller, Tolman, Pauli, Fock, etc. The present discussion will be confined to a few comments about 'rigid' bodies.

In Newtonian mechanics, a rigid body is defined to be a body in which the distance between any two particles making up the body is an invariant. If one of these two particles is acted on by a force, which has a component along the line joining the particles, such that one of the particles is moved, then according to Newtonian mechanics, this force must produce instantaneously a similar change in the position of the second particle, so that the separation of the two particles remains the same; otherwise the separation of the two particles would change and the body would not be a rigid body according to the classical definition. Therefore, a rigid body, in the classical sense, should be able to transmit forces between its constituent parts with infinite speed. According to the theory of special relativity, energy and momentum (and hence forces) cannot be transmitted with a velocity exceeding the velocity of light, so that according to the theory of special relativity there are no rigid bodies in the strict classical sense, and the concept of a rigid body is rarely introduced into the theory of special relativity. Sometimes the concept of a rigid body is introduced for purposes of discussion, for example, to quote McCrea:[21]

> We shall therefore now define *a rigid rod as one along which impulses are transmitted with speed c.*
> Since our rod is being treated as a one-dimensional body and since we are dealing only with longitudinal motion, no other condition is required. Moreover, since the theory permits the existence of no 'more rigid' body of this sort, there is no objection to adopting the term *rigid* in this sense.

At first sight it appears that this new definition does away with the normal ideas of everyday solid bodies such as a steel rod. Consider the propagation of longitudinal waves in such a rod. Let the rod be uniform, and let it be composed of a material of proper mass m_0 per unit length, and let the modulus of elasticity be λ. To quote McCrea again:

> According to both classical and relativistic mechanics the speed of longitudinal waves of small amplitude is then v where
>
> $$v^2 = \lambda/m_0$$

Since there is no dispersion, this is also the speed at which impulses of small amplitude traverse the rod. We have seen that relativity theory requires $v \leq c$ and hence we must have

$$\lambda \leq m_0c^2$$

Thus relativity theory imposes an upper limit m_0c^2 upon the modulus λ.

In practice λ/m_0 (which is equal to the ratio of Young's modulus to the density) is about 3×10^{11} c.g.s. units for steel, whilst c^2 is 9×10^{20} c.g.s. units, so that the relativistic upper limit is not of any practical significance, even for the most rigid substances known.

A classical rigid body is often used in classical physics as the ideal reference frame for measuring co-ordinates. Such a reference frame cannot be used in relativity theory since lengths are not invariant. On the other hand, according to the theory of special relativity, light moves in straight lines with the same speed in all inertial frames, so that straight lines can be defined in terms of light paths, and distances measured using the radar methods described in Section 3.9.

Generally, quantum mechanical problems have not been considered in the text. The reader is advised to have a thorough grounding in non-relativistic quantum mechanics before proceeding to relativistic quantum mechanics. It will be shown in Section 6.5 that the de Broglie relation $\lambda = h/p$ is consistent with the theory of special relativity. Schrödinger's equation is a non-relativistic equation and it can be shown that if the change of potential over the width of the wave pulse representing a particle is negligible, then Schrödinger's equation approximates closely to Newton's laws of motion. In order to obtain equations valid at relativistic energies one must start from the Hamiltonian given by eqn (5.48). The best known of these equations is the Dirac equation, which has been shown to be applicable to electrons (of spin $\frac{1}{2}$). The interested reader is referred to Mandl[22].

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PROBLEMS

Problem 5.1—Show how, from a consideration of the elastic collision of two spheres, the theory of Newtonian mechanics must be modified if it is to become consistent with the theory of special relativity.

Problem 5.2—Discuss, critically, the experimental evidence for the variation of mass with velocity.

Problem 5.3—Calculate the velocities of electrons accelerated through potential differences of (*a*) 10,000; (*b*) 100,000; and (*c*) 1,000,000 V. What is the ratio of the relativistic mass to the rest mass in each instance? (Hint: Use trigonometrical substitutions to calculate m/m_0 from u/c.)

Problem 5.4—Calculate the velocities of protons of kinetic energies (*a*) 10; (*b*) 100, and (*c*) 1,000 MeV.

Problem 5.5—Show that if a particle is highly relativistic ($\gamma = (1 - u^2/c^2)^{-\frac{1}{2}} \gg 1$), the fractional difference between c and v is approximately $\frac{1}{2}\{(m_0c^2)/E\}^2$, where E is the total energy.

Problem 5.6—Calculate the radii of curvature of electrons of velocities: (*a*) $0{\cdot}3c$; (*b*) $0{\cdot}8c$; (*c*) $0{\cdot}99c$, and (*d*) $0{\cdot}999c$ in a magnetic field of strength 1 Wb/m^2 (or 10,000 G).

Problem 5.7—Calculate the radius of curvature of a proton of velocity $0{\cdot}1c$ in a magnetic field of strength 1 Wb/m^2 (or 10,000 G).

Problem 5.8—Describe a β-ray spectrometer capable of focusing electrons of energies up to 3 MeV.

Problem 5.9—Discuss the physical principles underlying the following accelerators: (*a*) cyclotron; (*b*) synchrocyclotron; (*c*) electron synchrotron; (*d*) electron linear accelerator (Reference: McMillan[6]).

Problem 5.10—A particle of rest mass M moving with velocity u collides with a stationary particle of rest mass m. If the particles stick together show that the speed of the composite ball is equal to $u[\gamma M/(\gamma M + m)]$ where $\gamma = 1/(1 - u^2/c^2)^{\frac{1}{2}}$. (Hint: Proceed as in Section 5.7., but allow for the different masses of the colliding particles.)

Problem 5.11—A rocket propels itself rectilinearly through empty space by emitting pure radiation in the direction opposite to its motion. If V is its final velocity relative to its initial rest frame, prove that the ratio of the initial to the final rest-mass of the rocket is given by

$$\frac{M_i}{M_f} = \left[\frac{c + V}{c - V}\right]^{\frac{1}{2}} \qquad \text{(Rindler 1960)}$$

Problem 5.12—A hundred μ-mesons of rest mass 206 electron masses and energy 4·75 GeV are produced at an altitude of 30 km. If the mean life of μ-mesons at rest is $2{\cdot}2 \times 10^{-6}$ sec, calculate how many of the μ-mesons should reach sea level (*a*) allowing for time dilation, and (*b*) neglecting time dilation, if they travel vertically downwards without ionization loss.

Problem 5.13—The diameter of our galaxy is about 10^5 light years. How long does it take a proton (in the proton rest frame) to pass through the galaxy if its energy is (*a*) 10^{15} eV; (*b*) 10^{17} eV; (*c*) 10^{19} eV?

Problem 5.14—Calculate the amount of work in MeV that must be done to increase the velocity of an electron (*a*) to half the velocity of light (*b*) to three quarters the velocity of light. What is the ratio of the relativistic mass to the rest mass in each instance?

Problem 5.15—Through what potential must an electron fall if, according to Newtonian mechanics, its velocity is to equal the velocity of light? What speed does the electron actually acquire according to the theory of special relativity?

Problem 5.16—By what fraction does the mass of water increase, due to the increase in its thermal energy, it if is heated from 20°C to 50°C?

Problem 5.17—An incident photon of energy $h\nu$ is scattered through an angle of 180 degrees in a Compton-type collision with a stationary electron. Calculate the total energies after the collision of (*a*) the electron (*b*) the photon. [Hint: Apply the laws of conservation of total energy and of linear momentum in the lab.-system. Check your answer using eqns (5.90) and (5.93).]

Problem 5.18—Show that, if the electron recoils at an angle ϕ to the direction of motion of the incident photon of energy $h\nu$ in a Compton-type collision, then its kinetic energy is given by

$$T = h\nu \frac{2\alpha \cos^2\phi}{(1 + \alpha)^2 - \alpha^2 \cos^2\phi}$$

where $\alpha = \dfrac{h\nu}{m_0c^2}$. [Reference: Semat, *Foundations of Atomic and Nuclear Physics*, 3rd Edition, Appendix 7. Check the answer using eqn (5.93).]

Problem 5.19—A particle of rest mass m_1 having kinetic energy T_1 collides with a stationary particle of rest mass m_2. If m_1 is scattered through an angle of 90 degrees and has momentum p_1' after the collision, show that, after collision, m_2 has momentum $(p_1'^2 + 2T_1m_1 + T_1^2/c^2)^{\frac{1}{2}}$ and makes an angle θ_4 with the direction of the incident particle equal to $\tan^{-1}\{p_1'/(2T_1m_1 + T_1^2/c^2)^{\frac{1}{2}}\}$.

Problem 5.20—Calculate the total binding energy of one helium nucleus, if the mass of a hydrogen atom is 1·00814 a.m.u., that of a neutron 1·00898 a.m.u. and that of a helium atom 4·00388 a.m.u. (The masses are measured on the ^{16}O scale.)

Problem 5.21—Calculate the energy released in the following reactions:

(*a*) $${}^{7}_{3}\mathrm{Li} + {}^{1}_{1}\mathrm{H} \rightarrow 2\,{}^{4}_{2}\mathrm{He}$$

(*b*) $${}^{14}_{7}\mathrm{N} + {}^{4}_{2}\mathrm{He} \rightarrow {}^{17}_{8}\mathrm{O} + {}^{1}_{1}\mathrm{H}$$

(Hint: Look up accurate tables of a.m.u. and estimate the mass difference.)

Problem 5.22—The earth receives solar energy at the rate of $1{\cdot}35 \times 10^3$ W/m^2. If the distance of the earth from the sun is 150,000,000 km, find (*a*) the total mass lost by the sun per second; (*b*) the pressure of solar radiation on the earth assuming the earth is a black body.

Problem 5.23—Describe how the aberration of light from stars and the Doppler effect can be interpreted in terms of the momentum and energy transformations for photons.

Problem 5.24—Show that the rest mass of a particle is given by

$$m_0 = \frac{p^2c^2 - T^2}{2Tc^2}$$

where p is its momentum and T its kinetic energy. Calculate the rest mass of a particle if its momentum is 130 MeV/c when its kinetic energy is 50 MeV.

Problem 5.25—An unstable particle of rest mass M_0 and momentum $\mathbf{p}_0$ decays in flight into two particles of rest masses M_1 and M_2, momenta $\mathbf{p}_1$ and $\mathbf{p}_2$ and total energies E_1 and E_2, respectively. Show that

$$M_0^2c^4 = (M_1 + M_2)^2c^4 + 2E_1E_2 - 2M_1M_2c^4 - 2p_1p_2c^2 \cos\theta$$

where θ is the angle between the two decay particles. [Hint: Apply conservation of momentum and of mass-energy in the lab.-system.]

Problem 5.26—An unstable neutral particle decays into two charged particles of kinetic energies 190 MeV and 30 MeV and momenta 300 MeV/c and 240 MeV/c respectively. Determine the masses of the decay products. If the angle between the decay particles is 45 degrees, determine (*a*) the rest mass of the neutral particle, (*b*) its momentum, and (*c*) its kinetic energy. [Use $m_0 = (p^2c^2 - T^2)/2Tc^2$ to determine the masses of the decay products. Apply conservation of total energy and of momentum in the lab.-system to determine the energy and momentum of the parent particle; hence determine its mass.]

Problem 5.27—Determine the maximum kinetic energy in MeV, of the electron arising from μ-decay if

$$\mu \rightarrow \mathrm{e} + \nu + \bar{\nu}$$

The rest mass of the μ-meson is 206·8 electron masses, and the masses of the neutrino and antineutrino may be taken to be zero.

Problem 5.28—A π°-meson of rest mass m_0, velocity u, $\gamma = (1 - u^2/c^2)^{-\frac{1}{2}}$ decays in flight into two photons. If one of the photons is emitted at an angle θ to the direction of motion of the π°-meson in the laboratory system, show that its energy $h\nu$ is given by

$$h\nu = m_0c^2/\{2\gamma(1 - u\cos\theta/c)\}$$

Problem 5.29—A particle of momentum $\mathbf{p}_1$, rest mass m_1, is incident upon a stationary particle of rest mass m_2. Show that the velocity of the c.m.-system is equal to

$$\frac{\mathbf{p}_1c^2}{(E_1 + E_2)} = \frac{\mathbf{p}_1c^2}{\sqrt{(p_1^2c^2 + m_1^2c^4)} + m_2c^2}$$

Problem 5.30—Show that if a proton of velocity u, $\gamma = (1 - u^2/c^2)^{-\frac{1}{2}}$ collides with another proton, the velocity of the centre of mass is given by

$$u_{\text{c.m.}} = \frac{\gamma u}{(\gamma + 1)}$$

Show also that

$$\gamma_{\text{c.m.}} = \left[\frac{\gamma + 1}{2}\right]^{\frac{1}{2}}$$

[Hint: Use eqn (5.80).]

Problem 5.31—A particle of rest mass m_1, velocity $\mathbf{u}$ and total energy E_1 collides with a stationary particle of mass m_2. Show that in the extreme relativistic case, $u \simeq c$, the energy available in the c.m.-system is approximately $(2E_1m_2c^2)^{\frac{1}{2}}$. Show that the velocity of the c.m.-system is

$$\simeq c\left[1 - \frac{m_2c^2}{E_1}\right]$$

[Hint: Use eqn (5.121) to get the energy in the c.m.-system, and use (5.80) to determine the velocity of the c.m.-system.]

Problem 5.32—A photon collides with a stationary proton to produce a π^+-meson according to the reaction

$$\gamma + \text{p} \rightarrow \text{n} + \pi^+$$

Find the threshold frequency ν_0 for this process to take place. You may neglect the proton neutron mass difference. The rest mass of the π^+-meson is 273·2 electron masses.

If the mean lifetime of a π^+-meson in its rest frame is τ, obtain an expression for the mean life in the lab.-system of a π^+ created by a photon of frequency ν_0. [Hint: Derive and use eqn (5.121) to show that the threshold kinetic energy is 150 MeV corresponding to a frequency of $3{\cdot}6 \times 10^{22}$ c/sec. Calculate the velocity of the c.m.-system and calculate the dilation of lifetimes relative to the laboratory.]

Problem 5.33—A photon incident on a hydrogen target produces a π^+-meson (of rest mass 273·2 electron masses) according to the reaction

$$\gamma + \text{p} \rightarrow \text{n} + \pi^+$$

If the π^+-meson has a kinetic energy of 50 MeV, and is emitted at an angle of 90 degrees to the direction of motion of the incident photon, calculate the momentum of the π^+-meson in the lab.-system, and the energy of the incident γ-ray.

Problem 5.34—Show that, if total energy and momentum are to be conserved in pair production by a photon, the process cannot take place spontaneously in a vacuum.

If a photon strikes a stationary electron giving rise to an electron–positron pair as well as a recoil electron, show that the threshold for the reaction is $4m_0c^2$, where m_0 is the rest mass of an electron. [Hint: Assume for purposes of discussion that pair production takes place in a vacuum. The minimum electron momenta would be $\dfrac{h\nu}{2c}$, when both electrons were emitted in the direction of the incident photon. For this case the sums of the total energies of the electrons would be $2\left(m_0^2c^4 + \left(\dfrac{h\nu}{2c}\right)^2 c^2\right)^{\frac{1}{2}}$ which would be greater than $h\nu$, so that energy would not be conserved. For the second part of the question use eqn (5.124).]

Problem 5.35—Show that in positron annihilation at least two photons must be produced if linear momentum is to be conserved. Calculate the energies of the two photons, if the positron and electron are at rest when they annihilate each other.

Problem 5.36—Calculate the threshold kinetic energies, in MeV, for the following processes

(*a*) $\gamma + \mathrm{p} \rightarrow \mathrm{p} + \pi^0$

(*b*) $\pi^- + \mathrm{p} \rightarrow \pi^+ + \pi^- + \mathrm{n}$

(*c*) $\mathrm{p} + \mathrm{p} \rightarrow \mathrm{p} + \mathrm{p} + \mathrm{p} + \bar{\mathrm{p}}$

The rest masses of p, π^+ and π° are 1836, 273 and 264 electron masses respectively. [Hint: Derive and use eqn (5.124).]

Problem 5.37—A stationary nucleus of mass M is in an excited state of excitation energy E_0. It emits a photon in a transition to the ground state. Show, using relativistic equations, that the frequency of the photon, after allowing for recoil, is

$$\nu = \frac{E_0}{h}\left[1 - \frac{1}{2}\frac{E_0}{Mc^2}\right]$$

where h is Planck's constant. [Hint: Derive and use eqn (5.116).]

Problem 5.38—In the collision of a very high energy proton, of velocity u_0 and $\gamma = (1 - u_0^2/c^2)^{-\frac{1}{2}} \gg 1$ with a stationary proton, π-mesons are emitted symmetrically in the c.m.-system with velocity u', that is the number of π-mesons in the backward direction is equal to the number in the forward direction in the c.m.-system. Show that $\theta_{\frac{1}{2}}$, the angle in the lab.-system containing half the emitted π-mesons, is given by

$$\tan \theta_{\frac{1}{2}} = \frac{u'}{\gamma_{\mathrm{c.m.}} u_{\mathrm{c.m.}}} \simeq \frac{u' 2^{\frac{1}{2}}}{\gamma^{\frac{1}{2}} c}$$

[Hint: Consider a π-meson emitted at 90 degrees in the c.m.-system and find its direction in the lab.-system.]

Problem 5.39—A particle of charge q, rest mass m_0, is projected from the origin with velocity u_0 along the y axis. If there is a uniform electric field $\mathbf{E}$ parallel to the x axis, show that the orbit of the particle is a catenary given by $x = \dfrac{W_0}{qE}\cosh\dfrac{qEy}{p_0c}$, where p_0 is the initial momentum, $W_0 = (m_0^2c^4 + c^2p_0^2)^{\frac{1}{2}}$ is the initial total energy of the particle and $\mathbf{E}$ is the electric intensity vector. [Hint: The equations of motion are $\dot{p}_x = qE$ and $\dot{p}_y = 0$. Integration gives $p_x = qEt$ and $p_y = p_0$. Now if W is the total energy at a time t

$$\dot{x} = p_xc^2/W = qEtc^2/(W_0^2 + c^2q^2E^2t^2)^{\frac{1}{2}}$$

and

$$\dot{y} = p_yc^2/W = p_0c^2/(W_0^2 + c^2q^2E^2t^2)^{\frac{1}{2}}$$

Integrate with respect to t and eliminate t.]

Problem 5.40—What changes must be made to non-relativistic electron optics so that the theory can be applied to particles having velocities comparable to the velocity of light?

A large parallel plate capacitor, *in vacuo*, is charged to a potential difference of one million volts. The plates are 5 cm apart. Calculate how long it will take an electron starting from rest to travel from the negative to the positive plate. The charge and rest mass of the electron are $1{\cdot}60 \times 10^{-19}$ C and $9{\cdot}11 \times 10^{-31}$ kg respectively. The velocity of light is 3×10^8 m/sec. [Hint: Derive and use eqn (5.37).]

Problem 5.41—A particle P of rest mass m_0 is attracted towards the origin O of an inertial frame by a Newtonian force of magnitude $m_0\mu/r^2$ where $r = OP$ and μ is a constant. Prove that the motion of the particle is confined to a plane and that the differential equation of the orbit in terms of polar co-ordinates r, θ in the plane is

$$\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} + u\left(1 - \frac{\mu^2}{h^2c^2}\right) = \text{constant}$$

where $u = 1/r$, and $h = r^2\,\mathrm{d}\theta/\mathrm{d}s$ is the (constant) relativistic angular momentum per unit mass.

Show that by choice of initial line the solution of this equation may be written in the form

$$lu = 1 + e\cos k\theta$$

where $k = (1 - \mu^2/h^2c^2)^{\frac{1}{2}}$ and l, e are constants. Describe the motion briefly for the case $\mu \ll hc$, $e < 1$. (Exeter 1960) [Reference: W. H. McCrea, *Relativity Physics*, p. 29.]

6

GEOMETRICAL REPRESENTATION OF LORENTZ TRANSFORMATIONS AND THE USE OF 4-VECTORS

6.1. GEOMETRICAL REPRESENTATION OF THE GALILEAN TRANSFORMATIONS

The position of a particle in three dimensions can be specified by the x, y and z co-ordinates of the point. If the particle is moving, then these co-ordinates vary with time. It is often convenient to represent

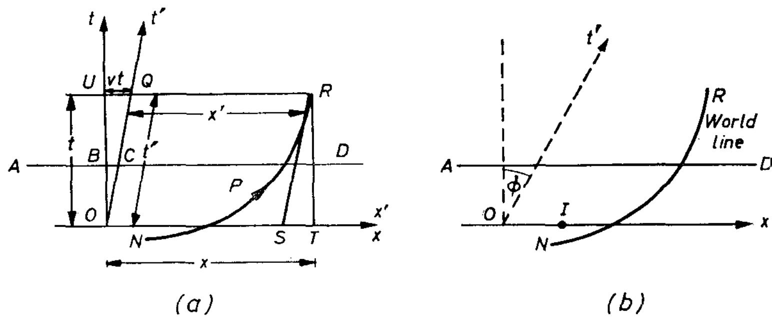

Figure 6.1. *(a) The displacement of the particle P along the x axis of an inertial frame Σ is represented by the curve NR in the Ox, Ot rectangular co-ordinate system. The displacement of P relative to Σ′ can be represented by the same curve NR, if Ox′ and Ot′ are chosen as oblique axes to represent the displacement of P relative to Σ′. A new length must be chosen to represent the unit of time in the oblique co-ordinate system. All that must be given are the 'world lines' such as NR, the origin O, the x axis, the length of the unit of x and the time calibration line AD which are all shown by continuous lines in (b). From this data a suitable co-ordinate system can be chosen*

this motion graphically. For simplicity we shall start by considering motion along the x axis only. Choose a rectangular co-ordinate system to represent the displacement of the particle relative to the inertial frame Σ. The displacement of a particle P along the x axis of Σ will be plotted against the time t as shown in *Figure 6.1.* Let the motion of a particle P relative to Σ be represented by the curve NR, and let the particle be at the position R having a displacement x at a time t.

Now consider the motion of the *same* particle from an inertial frame Σ', which is moving with uniform velocity v relative to Σ along the common x axis. If the origins of Σ and Σ' coincide at $t = t' = 0$, then according to the Galilean transformations one has

$$x' = x - vt; \quad t' = t$$

Since $x' = 0$ at the origin of Σ', the motion of the origin of Σ' relative to the origin of Σ is given by the line $x = vt$. This is represented by the line OQ in *Figure 6.1*. The lines Ox and OQ will now be used as *oblique* axes Ox' and Ot' respectively to represent the motion of the particle P relative to the inertial frame Σ'. We shall start by determining the x' and t' co-ordinates of the point R, without associating them in any way, at the moment, with the motion of the particle P. In order to get the x' co-ordinate of the point R, draw a line parallel to the Ot' axis to cut the Ox' axis at S; the x' co-ordinate of the point R can be represented by OS. Similarly, with a suitable choice of scale, the intercept of the line RU on the Ot' axis can be used to represent the t' co-ordinate of the point R in the oblique co-ordinate system. Relative to the *original* rectangular axes Ox, Ot, which were used to represent the displacement of the particle P relative to Σ,

$$UQ = vt \text{ and } UR = x$$

$$OS = QR = UR - UQ = x - vt$$

But, according to the Galilean transformations, the x' co-ordinate in Σ', corresponding to the x co-ordinate at a time t in Σ, is given by

$$x' = x - vt$$

Therefore, if the same scale is used along the Ox' axis as along the Ox axis, OS can be used to represent the displacement of the particle P relative to Σ', when the particle P is at the point R having a displacement x at a time t relative to Σ. The point R need not be displaced in the x' direction in order to represent the displacement of the particle P, relative to Σ'.

In Newtonian mechanics time is absolute so that time intervals must be measured to have the same numerical value in both Σ and Σ'. In *Figure 6.1* the length OQ is greater than the length OU. Hence, if the position of the point R relative to the Ox' axis is to represent the displacement of the particle P at a time $t' = t$ in Σ', then a new time scale must be chosen for the oblique co-ordinate system Ox', Ot'. Draw the line $ABCD$ corresponding to the time $t = 1$ in the original rectangular co-ordinate system Ox, Ot. The length OB represents the unit of time in Σ. If OC is chosen as the

new unit of time in the oblique co-ordinate system Ox', Ot', then the time t' represented by the point R in the Ox', Ot' co-ordinate system is equal to the time t represented by the point R in the co-ordinate system Ox, Ot. Thus using the oblique co-ordinate system Ox', Ot' to represent the displacement of the particle P relative to Σ', and using a new time scale, the position of the point R need not be changed in *Figure 6.1*, and the curve NR can be used to represent the displacement of the particle P relative to both Σ and Σ'.

An oblique co-ordinate system, with the same x axis as Σ, but a different time axis and a different time scale, can be chosen to represent the displacement of the particle P relative to any other inertial frame which is moving with uniform velocity relative to Σ along the x axis, whose origin coincides with the origin of Σ at $t = 0$. We need not have started with a rectangular co-ordinate system, but could have started with oblique axes. The unit of time is cut off on all the time axes by the line AD.

If the particle P moves in two dimensions in the xy plane, the motion can be represented in a 3-dimensional co-ordinate system x, y and t. For motion in three dimensions one would have to use the four dimensions x, y, z and t to represent the displacement. It should be emphasized that there is nothing more unusual about using the four dimensions x, y, z and t to represent the motion of the particle in the theory of special relativity, than in using the same number of dimensions to represent the same co-ordinates and time in Newtonian mechanics. Provided the length of the unit of time is chosen appropriately, the curve NR, which represents the displacement of the particle P, is the same for all the co-ordinate systems. It is called the 'world line' of the particle P. If the particle P moves in a straight line with uniform velocity relative to an inertial frame, the 'world line' is a straight line. If the particle is accelerated its 'world line' is curved. If the 'world line' NR and the origin O are given, then for motion in the x direction, one only needs OI the unit of length in the x direction and the line AD as shown in *Figure 6.1*(b). The unit of time for any co-ordinate system can be calculated, if the line AD is given. The x axis is common to all the possible co-ordinate systems representing the various inertial frames moving with uniform velocities relative to each other and whose origins coincide at $t = 0$. In order to select a suitable time axis draw *any* line from O to cut the line AD as shown in Figure *6.1*(b). The unit of time for this co-ordinate system is represented by the distance from O to the point of intersection of AD and Ot'. The velocity of this inertial frame relative to Σ is given by $v = \tan \phi$.

6.2. GEOMETRICAL REPRESENTATION OF THE LORENTZ TRANSFORMATIONS*

We shall again start by considering the motion of a particle P along the x axis of the inertial frame Σ, and start with a rectangular Cartesian co-ordinate system to represent the displacement of the particle P relative to Σ. Instead of plotting x against t, x will be plotted against ct, where c is the velocity of light. This gives the

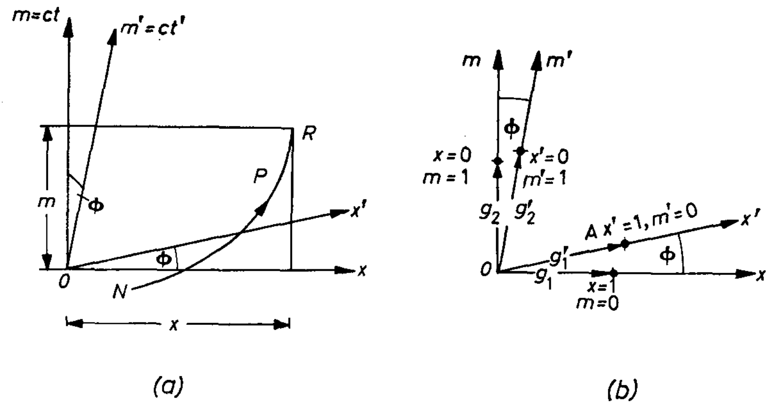

Figure 6.2. (a) The displacement of the particle P relative to the inertial frame Σ is represented by the curve NR in the Ox, Om rectangular co-ordinate system. The lines Ox′ and Om′ which are inclined at angles $\phi = \tan^{-1} v/c$ to the Ox and Om axes respectively can be used as oblique axes to represent the displacement of the particle P relative to the inertial frame Σ'. If new scales (or gauges) are chosen for the units of x′ and m′, then the displacement of the particle P relative to Σ' can also be represented by the same curve NR. (b) The gauges appropriate to Σ and Σ'

ordinate axis the dimensions of a length. Let ct be denoted by m. This diagram is called a *space-time* or *Minkowski* diagram (cf. Section 3.9 and Appendix 6*h*).

Let the motion of the particle P relative to Σ be represented by the curve NR, as shown in *Figure 6.2(a)*. We proceed now to see if the same curve NR can be used to represent the displacement of the particle P relative to an inertial frame Σ', which is moving with uniform velocity v along the x axis relative to Σ. From the Lorentz transformations

$$x' = \gamma(x - vt) \tag{6.1}$$

$$t' = \gamma(t - vx/c^2) \tag{6.2}$$

* Some readers may prefer to proceed to Section 6.3 before reading Section 6.2. The approach adopted in 6.3 is based on the co-ordinates x, y, z, ict. The latter method is a complete alternative to the method developed in Section 6.2 and is mathematically the simpler of the two.

Relative to the rectangular axes Ox, Om the lines $t' = m' = 0$ and $x' = 0$ are given by the equations $m = ct = vx/c$ and $x = vt = vm/c$ respectively. These lines are represented by Ox' and Om' in *Figure 6.2(a)*. The line Ox' makes an angle ϕ with the Ox axis and the line Om' makes an angle ϕ with the Om axis and an angle of $(\pi/2 - \phi)$ with the Ox axis, as shown in *Figure 6.2(a)*, where $\tan \phi = v/c$. The same plan will be followed as in Section 6.1, and Ox' and Om' will be used as oblique axes to represent the displacement of the particle P relative to Σ'. Unlike Newtonian mechanics, in addition to the time axis, the direction of the co-ordinate axis is also changed, so that one must be prepared to change the unit of length as well as the unit of time in the Ox', Om' co-ordinate system. The unit of length in the Ox', Om' co-ordinate system is given by the distance of the point $x' = 1$, $m' = ct' = 0$, from the origin as shown in *Figure 6.2(b)*. Relative to the original Ox, Om rectangular co-ordinate system, this point has co-ordinates

$$x = \gamma(x' + vt') = \gamma(1 + v0) = \gamma \tag{6.3}$$

$$m = ct = \gamma\left(ct' + \frac{vx'}{c}\right) = \gamma\frac{v}{c} \tag{6.4}$$

The distance of this point from the origin, relative to Σ, is equal to

$$g_1' = \sqrt{x^2 + m^2} = \sqrt{x^2 + c^2t^2}$$

$$g_1' = \gamma\sqrt{1 + v^2/c^2} = \sqrt{\frac{(1 + v^2/c^2)}{(1 - v^2/c^2)}} \tag{6.5}$$

If lengths along the Ox' axis are measured using the scale (or ruler) appropriate to the Ox, Om rectangular co-ordinate system, then in order to convert these measured lengths into lengths appropriate to the Ox', Om' co-ordinate system, the measured lengths must be divided by g_1'. The quantity g_1' is sometimes called the gauge of the co-ordinate system representing the displacement of the particle P relative to Σ'.

Similarly, the distance of the point $m' = ct' = 1$; $x' = 0$ from the origin of *Figure 6.2(b)* gives the length of the unit of m' relative to the rectangular co-ordinate system Ox, Om. Using the Lorentz transformations corresponding to $m' = 1$, $x' = 0$ in the oblique axes Ox', Om', in the rectangular co-ordinate system Ox, Om one has $x = \gamma v/c$ and $m = \gamma$ so that

$$g_2' = \gamma\sqrt{1 + v^2/c^2} \tag{6.6}$$

Lengths measured along the Om' axis using the scale (or ruler) appropriate to the rectangular axes Ox, Om must be divided by g_2' in order to obtain the values of m' appropriate to the oblique axes Ox', Om'.

The co-ordinates of the point R shown in *Figure 6.2(a)* will now be calculated relative to the oblique axes Ox', Om'. For the present the point R will not be associated in any way with the displacement of the particle P relative to Σ'. Draw lines through R parallel to the Ox' and Om' axes to cut lengths OT and OW on the Om' and Ox' axes respectively as shown in *Figure 6.3.* Let these be measured,

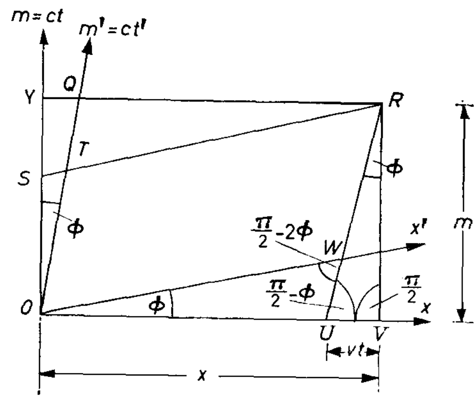

Figure 6.3. A more detailed form of Figure 6.2(a)

initially, using the scales (or gauges) of the rectangular co-ordinate system Ox, Om. Now, in *Figure 6.3*

$$OV = x;\ UV = m \tan \phi = vm/c = vt$$

Hence,

$$OU = OV - UV = (x - vt)$$

In the triangle OUW, the angles $\widehat{WOU}$, $\widehat{OUW}$ and $\widehat{OWU}$ are equal to ϕ, $\pi/2 + \phi$ and $\pi/2 - 2\phi$ respectively as shown in *Figure 6.3.* Hence

$$\frac{OW}{\sin(\pi/2 + \phi)} = \frac{OU}{\sin(\pi/2 - 2\phi)} = \frac{WU}{\sin \phi} \tag{6.7}$$

Therefore

$$OW = \frac{OU \cos \phi}{\cos 2\phi} = \frac{(x - vt) \cos \phi}{\cos 2\phi} \tag{6.8}$$

Since,

$$\tan \phi = v/c,\ \cos \phi = \frac{c}{\sqrt{c^2 + v^2}} = \frac{1}{\sqrt{1 + v^2/c^2}}$$

$$\cos 2\phi = 2 \cos^2 \phi - 1 = \frac{2}{(1 + v^2/c^2)} - 1 = \frac{(1 - v^2/c^2)}{(1 + v^2/c^2)}$$

Substituting in eqn (6.8) one obtains

$$OW = \frac{(x - vt)/(1 + v^2/c^2)^{+\frac{1}{2}}}{(1 - v^2/c^2)/(1 + v^2/c^2)} = \frac{(x - v\,t)\,(1 + v^2/c^2)^{\frac{1}{2}}}{(1 - v^2/c^2)} \tag{6.9}$$

This is the length of OW measured using the scale appropriate to the rectangular co-ordinate system Ox, Om. In the oblique co-ordinate system Ox', Om' the unit of length must be changed and the length OW given by eqn (6.9) must be divided by the gauge

$$g_1' = \gamma(1 + v^2/c^2)^{\frac{1}{2}}$$

Hence, relative to Ox', Om', the distance OW represents a length equal to

$$\frac{(x - vt)(1 + v^2/c^2)^{\frac{1}{2}}}{(1 - v^2/c^2)} \div \frac{(1 + v^2/c^2)^{\frac{1}{2}}}{(1 - v^2/c^2)^{\frac{1}{2}}} = \frac{(x - vt)}{\sqrt{1 - v^2/c^2}}$$

According to the Lorentz transformations, when the particle P is at the point x at a time t relative to Σ, its displacement relative to Σ' is equal to

$$x' = \gamma(x - vt)$$

Hence, using the new choice of gauge given by eqn (6.5), OW is equal to the x' co-ordinate of the particle P in Σ', corresponding to the event at R having a co-ordinate x at a time t in Σ.

In *Figure 6.3* using the scales appropriate to the rectangular axes Ox, Om,

$$OT = WR = RU - WU = \frac{m}{\cos\phi} - WU = \frac{ct}{\cos\phi} - WU$$

From eqn (6.7),

$$WU = \frac{OU\sin\phi}{\sin\left(\frac{\pi}{2} - 2\phi\right)} = \frac{OU\sin\phi}{\cos 2\phi}$$

$$= \frac{(x - vt)\dfrac{v}{(c^2 + v^2)^{\frac{1}{2}}}}{\dfrac{(1 - v^2/c^2)}{(1 + v^2/c^2)}} = \frac{(x - vt)\,\dfrac{v}{c}\,(1 + v^2/c^2)^{\frac{1}{2}}}{(1 - v^2/c^2)}$$

$$OT = RU - WU = ct(1 + v^2/c^2)^{\frac{1}{2}} - \frac{(x - vt)\,\dfrac{v}{c}\,(1 + v^2/c^2)^{\frac{1}{2}}}{(1 - v^2/c^2)}$$

$$OT = \frac{(1 + v^2/c^2)^{\frac{1}{2}}}{(1 - v^2/c^2)}\left(ct - \frac{vx}{c}\right) \tag{6.10}$$

This is the length of OT measured using the scale appropriate to the rectangular co-ordinate system Ox, Om. In order to find the length this distance represents in the oblique co-ordinate system Ox', Om', eqn (6.10) must be divided by the appropriate gauge, which is given by

$$g_2' = \gamma\sqrt{1 + v^2/c^2} \tag{6.6}$$

Dividing the right-hand side of eqn (6.10) by g_2', one obtains

$$\frac{(1 + v^2/c^2)^{\frac{1}{2}}}{(1 - v^2/c^2)}\left(ct - \frac{vx}{c}\right) \div \gamma\sqrt{1 + v^2/c^2} = \gamma\left(ct - \frac{vx}{c}\right)$$

Relative to the oblique co-ordinate system Ox', Om' the distance OT represents a length $\gamma(ct - vx/c)$. According to the Lorentz transformations corresponding to the time t, when the particle P has a displacement x in Σ, one has in Σ'

$$ct' = \gamma\left(ct - \frac{vx}{c}\right)$$

Hence, with the new choice of gauge for m', the m' co-ordinate of the point R is exactly equal to the value predicted by the Lorentz transformations for the particle P at the time t when it has a displacement x in Σ. Thus by choosing Ox' and Om' as oblique axes to represent the displacement x' of the particle P at a time t' in Σ', corresponding to the displacement x at a time t in Σ, and using the new gauges given by eqns (6.5) and (6.6) respectively, the position of the point R on the graph shown in *Figure 6.2(a)* need not be changed. This conclusion holds whatever the relative velocity between the two inertial frames, provided the relative velocity is constant. If the displacement of the particle P relative to the inertial frame Σ is represented by the curve NR in the Ox, Om rectangular co-ordinate system as shown in *Figure 6.2(a)*, then provided the scales (or gauges) for x' and ct' are chosen according to eqns (6.5) and (6.6), then the displacement of the particle P relative to the inertial frame Σ' is represented by the same curve NR in the oblique co-ordinate system Ox', Om'. This holds for all the inertial frames moving with uniform velocity along the x axis relative to Σ, whose origins coincide with each other at $t = 0$. Once the axes representing one inertial frame are chosen, the path NR is fixed. The path NR is called the '*world line*' of the particle P in the two dimensional x, ct 'world'. We started with rectangular axes in order to simplify the discussion. This was not necessary; one could have started with oblique axes.

Some of the geometrical relationships between the scales (gauges) corresponding to the various inertial frames are now discussed.

For the inertial frame Σ' the length OA' from the origin to the point A' (having co-ordinates $x' = 1$, $m' = 0$) in *Figure 6.4*(*a*) gives the length of the unit of x' along the Ox' axis. According to the Lorentz transformations, relative to the rectangular co-ordinate system Ox, Om, used to represent the displacement of the particle P relative to Σ, the point A' must have the co-ordinates $x = \gamma$, $m = ct = \gamma(v/c)$. For the point A'

$$x^2 - m^2 = x^2 - c^2t^2 = \gamma^2 - \gamma^2v^2/c^2 = \gamma^2(1 - v^2/c^2) = +1$$

The expression $(x^2 - m^2)$ is independent of v, the relative velocity of Σ' and Σ. *Relative to the rectangular co-ordinate system Ox, Om* the

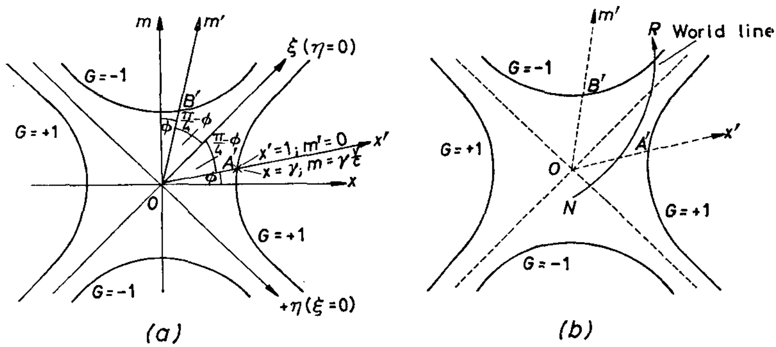

Figure 6.4. In (a) the calibration hyperbolae are shown. These enable the unit of length and time in any co-ordinate system such as Ox′, Om′ to be determined. The unit of x′ for the Ox′ axis shown in (a) is equal to OA′, and the unit of m′ is equal to OB′. The positions of the 'world lines' relative to the calibration hyperbolae are independent of the choice of any particular co-ordinate system. All that must be given are the 'world lines' and the calibration hyperbolae shown by full lines in (b). The asymptotes can then be drawn and a co-ordinate system such as Ox′, Om′ chosen

point A', whose distance from the origin gives the unit of length for the co-ordinate system used to represent the displacement relative to Σ', always lies on the curve

$$x^2 - m^2 = x^2 - c^2t^2 = G = +1 \tag{6.11}$$

whatever the value of v. If the relative velocity between Σ' and Σ were different, the position of the point A' which has co-ordinates $x = \gamma$, $m = ct = \gamma(v/c)$, would vary since v and γ would vary, but A' would always be on the curve $x^2 - m^2 = +1$ relative to Σ. Eqn (6.11) is the equation of a hyperbola. It is illustrated in *Figure 6.4*(*a*). The unit of length for any oblique co-ordinate system Ox', Om' is given by the distance from the origin to the point of intersection of the Ox' axis and the hyperbola corresponding to $G = +1$.

The unit of m' is given by the length OB', where B' corresponds to the point $x' = 0$; $m' = ct' = 1$ in *Figure 6.4*(*a*). According to the Lorentz transformations the x and m co-ordinates of B' are $\gamma(v/c)$ and γ respectively. For the point B', relative to the rectangular co-ordinate system Ox, Om one has

$$x^2 - m^2 = x^2 - c^2t^2 = \gamma^2 \frac{v^2}{c^2} - \gamma^2 = -1$$

Thus B' always lies on the hyperbola

$$x^2 - m^2 = G = -1 \tag{6.12}$$

whatever the value of v. The unit of m' in Σ' is given by the distance from the origin to the point of intersection of the Om' axis with the hyperbola given by eqn (6.12). The two sets of hyperbolae, given by eqns (6.11) and (6.12) respectively, are known as the calibration hyperbolae.

If new co-ordinates $\eta = x - m$ and $\xi = x + m$ are introduced, then the hyperbolae take the form $\xi\eta = \pm 1$. The lines $\eta = 0$, $\xi = 0$ are the asymptotes to the hyperbolae, the line $\eta = 0$ corresponding to $x = +ct$ and the line $\xi = 0$ corresponding to $x = -ct$. These are the 'world lines' of light signals leaving the origin in the positive and negative x directions respectively at a time $t = 0$. These particular 'world lines' were introduced in Section 3.12 where it was shown that they separate events which can and cannot be causally connected with the event at the origin at the time $t = 0$ (compare *Figure 3.14*). The line $x = +ct = +m$ bisects the angle between the Ox and Om axes. Since the Ox' axis is inclined at the same angle ϕ to the Ox axis as the Om' axis is inclined to the Om axis, the line $x = +ct$ corresponding to $\eta = 0$ also bisects the angle between the Ox' and Om' axes.

In order to calculate the calibration curves we started with rectangular axes; this was not necessary, we could have started with oblique axes. All that need be given are the calibration hyperbolae and the world lines as shown in *Figure 6.4*(*b*). The asymptotes to the calibration hyperbolae can then be drawn. In order to select a suitable co-ordinate system, one need only draw any straight line from the origin to cut the hyperbola $x^2 - m^2 = G = +1$, say at the point A' as shown in *Figure 6.4*(*b*); this line serves as the Ox' axis and OA' is the unit of length for this particular co-ordinate system. To obtain the corresponding Om' axis draw a straight line, from the origin, on the other side of the asymptote $\eta = 0$, and at the same inclination to it as the Ox' axis has, as shown in *Figure 6.4*(*b*). This line is the appropriate Om' axis. The unit of m' is equal to

OB', where B' is the point of intersection of the Om' axis with the calibration hyperbola $x^2 - m^2 = G = -1$. Any such oblique co-ordinate system, obtained in this way, can be used to describe the course of nature. The world lines are the same for all of them. According to the principle of relativity the laws of physics should be the same for all these inertial frames of reference.

If the particle P moves in the xy plane one would have to use 3 dimensions to represent the displacement of the particle at the time t. For motion in 3 dimensions one would have to use 4 dimensions corresponding to $x, y, z, m = ct$. Since $y' = y$ and $z' = z$ no change of gauge is necessary in the y and z directions and the Oy and Oy' axes are coincident, and so are the Oz and Oz' axes. Minkowski (1908) suggested that the 4-dimensional x, y, z, ct space represents the '*world*'. In this 'world' *events* or 'world points' are specified by the four co-ordinates x, y, z, ct. The motions of particles are represented by 'world lines'. The motion of a particle moving with uniform velocity is represented by a straight 'world line', accelerated motion is represented by a curved 'world line'. These 'world points' and 'world lines' are the same for all the co-ordinate systems used to represent the displacements of particles in the various inertial frames moving with uniform velocity relative to each other and whose origins coincide at $t = 0$. In Newtonian mechanics the time interval between events and the lengths of objects are absolute. According to the theory of special relativity the numerical values of lengths and time intervals are not absolute, but depend on the inertial reference frame chosen to represent the motions. However, it was shown in Section 3.12 that the interval between 'events' is absolute, that is, has the same numerical value in all inertial frames of reference. The interval δs was defined by the relation

$$\delta s^2 = \delta x^2 + \delta y^2 + \delta z^2 - c^2\,\delta t^2 \tag{3.39}$$

Thus, though the spatial and temporal intervals between two events are not absolute, according to the theory of special relativity a certain combination of the two, namely the interval between the events, is absolute. So are the 'world points' and 'world lines'. This is probably the way one should interpret the opening remarks which Minkowski made in his address delivered to the Eightieth Assembly of German Natural Scientists and Physicians at Cologne, on 21st September, 1908. In this address Minkowski introduced the geometrical interpretation of space and time for the first time. Minkowski said:

> The views of space and time which I wish to lay before you have sprung from the soil of experimental physics, and therein lies their strength.

They are radical. Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality.

In the 4-dimensional x, y, z, ct 'world' the kinematics of particles can be developed as a 4-dimensional geometry. There is an alternative approach in which the four variables x, y, z, ict are used, where $i = \sqrt{-1}$. This approach is mathematically simpler and will be developed in Section 6.3. The development of relativistic

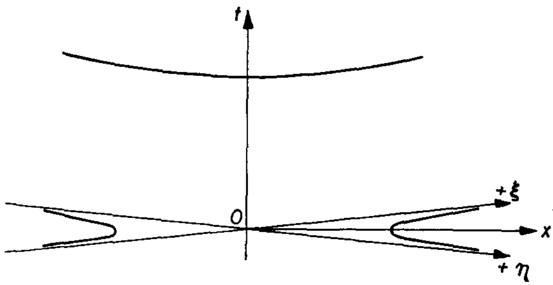

Figure 6.5. x is plotted against t and not against m = ct as was done in Figure 6.4. As a result the diagram looks more like the corresponding case for Newtonian mechanics shown in Figure 6.1. The choice for the direction of the x′ axis appears more restricted in Figure 6.5 than in Figure 6.4

kinematics and relativistic mechanics in terms of 4-vectors will be discussed in Section 6.4.

In Newtonian mechanics there is no choice of direction for the x' axis, once one set of axes is chosen. It appears at first sight, from *Figure 6.4(b)*, that according to the theory of special relativity there is a very wide choice for the direction of the x' axis. The choice is exaggerated, since we chose to plot x against ct rather than against t, as was done in *Figure 6.1*, Section 6.1. Consequently, the ordinates in *Figure 6.4(b)* were magnified by a factor c compared with the ordinates in *Figure 6.1*. If x had been plotted against t, *Figure 6.4* would be more like *Figure 6.5*, which is still enormously exaggerated. The choice of direction for the x' axis now appears very small. As c tends to infinity the equations of the theory of special relativity go over into the equations of Newtonian mechanics. This is true in the present case since, if c tends to infinity, the hyperbola corresponding to $G = +1$ tends to coincide with the x axis and the hyperbola corresponding to $G = -1$ tends to a straight line parallel to the x axis in agreement with *Figure 6.1(b)*.

As examples of the use of the methods developed in this section the phenomena of length contraction and time dilation will be discussed. Draw the calibration curve $G = 1$ as shown in *Figure 6.6(a)*. Let a rod of unit length be at rest on the x axis of an inertial frame Σ, which will be represented by the rectangular co-ordinate system Ox, Om; the choice of rectangular axes is not essential. The positions of the ends of the rod are given by the 'world lines' Om and AA' respectively as shown in *Figure 6.6(a)*. These 'world lines' are the same for all the possible co-ordinate systems. Let the axes

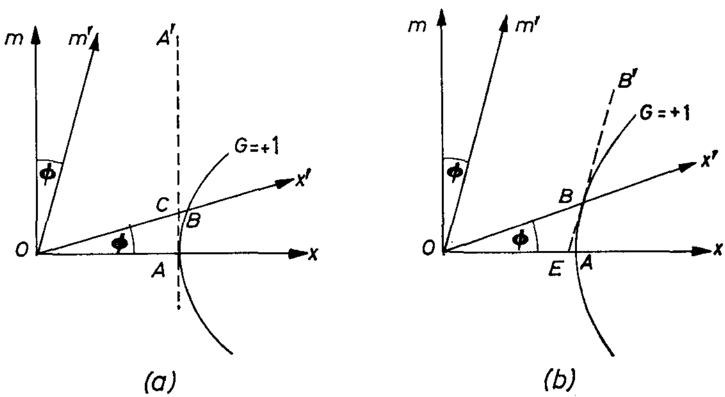

Figure 6.6. The phenomenon of length contraction. In (a) a rod of unit length is at rest in Σ, and the 'world lines' of its ends are represented by Om and AA'. In (b) the rod is at rest in Σ' and the 'world lines' corresponding to the ends of the rod are represented by Om' and BB'. The 'world line' BB' is a tangent to the calibration hyperbola at B

appropriate to an inertial frame Σ', moving with velocity v relative to Σ, be Ox' and Om', such that $\tan\phi = v/c$, as shown in *Figure 6.6(a)*. The unit of x' for the oblique Ox', Om' axes is equal to the length OB. In order to measure the length of the rod in Σ', the positions of the ends of the rod must be observed at the same time t' in Σ', say at $t' = 0$. The distance between the ends of the rod at $t' = 0$ is given by the distance between the points of intersection of the line $m' = 0$ with the two invariant 'world lines', that is by the length OC. But OC is less than OB the unit of length in Σ', so that the measured length of the rod in Σ' is less than unity, illustrating length contraction. [Actually in Σ', OC represents a length $x' = OC/g_1' = \sec\phi/g_1' = \sqrt{1 - v^2/c^2}$.]

If a rod of unit length is at rest along the x' axis of Σ', then the world lines of the ends of the rod are parallel to the Om' axis as shown in *Figure 6.6(b)* and are represented by the world lines Om'

and BB' in *Figure 6.6(b)*. The distance between the world lines in Σ at the time $t = 0$ is given by OE which is less than OA, the unit of length in Σ. This illustrates that length contraction is reciprocal.

Now time dilation is illustrated. Draw the calibration curve $G = -1$ as shown in *Figure 6.7(a)*. Let a clock at rest at the point $x = y = z = 0$ in Σ emit ticks at $t = 0$ and $t = 1/c$ corresponding to $m = 0$ and $m = 1$ respectively. The 'world points' corresponding to these events are at O and D respectively in *Figure 6.7(a)*. In order to measure the co-ordinates and times of these events relative to Σ',

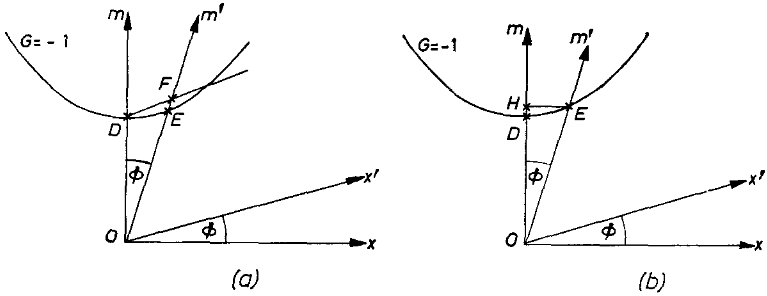

Figure 6.7. The phenomenon of time dilation. (a) Two events occurring at the point $x = 0$ in Σ at times $m = 0$ and $m = 1$ are represented by the 'world points' O and D respectively. In the triangle ODF, the angles ODF and OFD are $\pi/2 + \phi$ and $\pi/2 - 2\phi$ respectively, so that, $OF = OD \cos\phi \sec 2\phi$, where $\tan\phi = v/c$. (b) Two events occurring at the origin of Σ' at $m' = 0$ and $m' = 1$ are represented by the 'world points' O and E respectively

draw the Ox' and Om' axes appropriate to the relative velocity v, such that $\tan\phi = v/c$. The unit of m' in Σ' is given by OE. In order to obtain the m' co-ordinates of the event at D, draw a line through D parallel to the Ox' axis to cut the Om' axis at F; thus OF represents the difference in $m' = ct'$ values between the events at O and D in the inertial frame Σ'. Now the unit of m' in Σ' is given by OE. Since in *Figure 6.7(a)* $OF > OE$ the time interval between the two events, which occur at the point $x = y = z = 0$ in Σ, is longer in Σ' than in Σ illustrating time dilation. [Actually, OF represents a value of m' in Σ' equal to $OF/g_2' = \cos\phi \sec 2\phi/g_2' = \gamma$.] Let events corresponding to $x' = 0$, $m' = 0$ and $x' = 0$, $m' = 1$ be at O and E respectively as shown in *Figure 6.7(b)*. To obtain the m co-ordinate of the event at E draw a line parallel to the Ox axis to cut the m axis at H. The unit of $m = ct$ in Σ is given by OD. Since $OH > OD$, the time interval is now longer in Σ than in Σ' illustrating that the phenomenon of time dilation is reciprocal.

6.3. THE USE OF THE COMPLEX VARIABLE $X_4 = \mathrm{i}ct$

In Section 6.2 only real variables were introduced. There is an alternative approach in which instead of time or $m = ct$, the variable $X_4 = \mathrm{i}ct$ is introduced, where $\mathrm{i} = \sqrt{-1}$. Let new variables be introduced such that in Σ

$$X_1 = x; \quad X_2 = y; \quad X_3 = z; \quad X_4 = \mathrm{i}ct$$

Let the corresponding quantites in Σ' be

$$X_1' = x'; \quad X_2' = y'; \quad X_3' = z'; \quad X_4' = \mathrm{i}ct'$$

From the Lorentz transformations

$$x' = \gamma(x - vt)$$

Hence
$$X_1' = \gamma\left(X_1 - \frac{vX_4}{\mathrm{i}c}\right) = \gamma\left(X_1 + \mathrm{i}\frac{v}{c}X_4\right) \tag{6.13}$$

$$X_2' = X_2 \tag{6.14}$$

$$X_3' = X_3 \tag{6.15}$$

Now $\quad t' = \gamma(t - vx/c^2), \quad$ or $\quad \mathrm{i}ct' = \gamma\left(\mathrm{i}ct - \mathrm{i}\dfrac{v}{c}x\right)$

Hence,

$$X_4' = \gamma\left(X_4 - \mathrm{i}\frac{v}{c}X_1\right) \tag{6.16}$$

Eqns (6.13) and (6.16) can be rewritten in the form

$$X_1' = X_1 \cos\phi + X_4 \sin\phi \tag{6.17}$$

$$X_4' = -X_1 \sin\phi + X_4 \cos\phi \tag{6.18}$$

where

$$\tan\phi = \mathrm{i}v/c; \quad \cos\phi = \gamma; \quad \sin\phi = \mathrm{i}\gamma\frac{v}{c} \tag{6.19}$$

We shall again start by considering motion in the x direction only and X_4 will be plotted against X_1 using a rectangular Cartesian co-ordinate system as shown in *Figures 6.8*(*a*) and (*b*). Let the motion of the particle P relative to the inertial frame Σ be represented by the curve NR. We shall proceed now to see how the motion of P relative to Σ' can be represented. The X_1' axis of Σ' must be represented by the line $X_4' = 0$. It follows from eqn (6.18) that the line $X_4' = 0$ is given by $X_4/X_1 = \tan\phi$ relative to the OX_1, OX_4 co-ordinate system. This represents a straight line making an

angle ϕ with the X_1 axis. Similarly, the X_4' axis in Σ' must be represented by the line $X_1' = 0$. From eqn (6.17) it follows that, relative to the OX_1, OX_4 co-ordinate system, the line $X_1' = 0$ is represented by the line $X_1/X_4 = -\tan\phi$; this line makes an angle ϕ with the X_4 axis and an angle $(\pi/2 + \phi)$ with the X_1 axis as shown in *Figure 6.8*. In the present case the X_1 and X_4 axes are both rotated in the same direction when one transforms from Σ to Σ', and the axes always remain at right angles to each other.

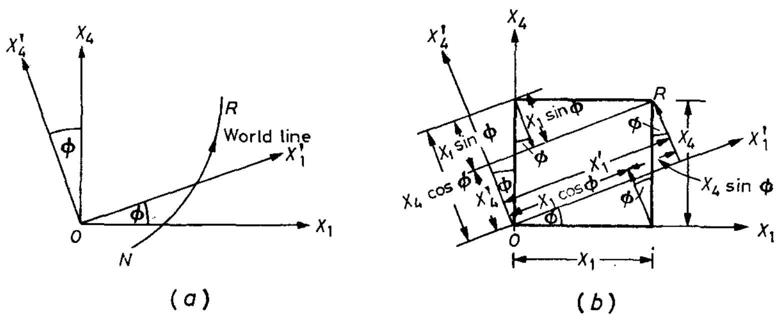

Figure 6.8. (a) NR is the 'world line' in the $X_1 = x$, $X_4 = ict$ space. A Lorentz transformation corresponds to the rotation of the X_1, X_4 axes through the same angle ϕ. (b) The relationships between the co-ordinates of 'the world point' R relative to the new axes X_1' X_4' and relative to the old axes X_1 X_4

If there is no change of gauge (or scale), then the co-ordinates of the point R relative to the OX_1', OX_4' axes are given by

$$X_1' = X_1 \cos\phi + X_4 \sin\phi$$

$$X_4' = -X_1 \sin\phi + X_4 \cos\phi$$

as illustrated in *Figure 6.8(b)*. These components are precisely the same as those predicted by the Lorentz transformations for the X_1' and X_4' co-ordinates of an event which has co-ordinates X_1 and X_4 in Σ. Therefore the point R can represent the displacement of the particle P relative to both Σ' and Σ without having to change the lengths of the units of either X_1' or X_4'. The 'world line' of the particle P is again an invariant. The Lorentz transformation corresponds to a rotation of the axes in the X_1X_4 plane.

The displacement OR in the X_1X_4 plane has the same transformation properties as the displacement vector in ordinary two dimensional co-ordinate geometry. For this reason it is often stated that the X_1 and X_4 co-ordinates of the point P are the co-ordinates of a vector in the X_1X_4 plane. If the particle P moves in three dimensions

then the four dimensions X_1, X_2, X_3, X_4 are necessary to specify its displacement at a given time. Since according to the Lorentz transformations $X_2' = X_2$ and $X_3' = X_3$, the Lorentz transformations can still be represented by a rotation of the axes in the X_1X_4 plane. The four co-ordinates of the point R are said to form the components of a 4-vector. By introducing the co-ordinate $X_4 = ict$ the geometry of the four-dimensional $X_1X_2X_3X_4$ space has become 'Euclidean'. It should be stressed that this is a purely formal

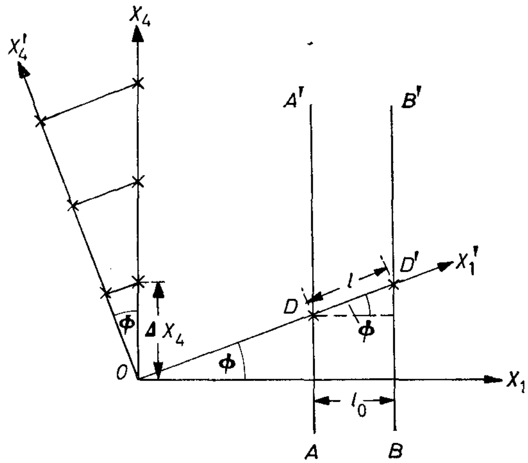

Figure 6.9. The illustration of the phenomena of time dilation and length contraction using the properties of the X_1X_4 space

representation, since the co-ordinate X_4 is imaginary and the rotation in the X_1X_4 plane is through an imaginary angle given by $\tan \phi = iv/c$. The use of complex numbers often simplifies the algebra. No deep metaphysical significance should be attached to the presence of $\sqrt{-1}$ in some of the equations of the theory of special relativity. It is introduced simply for reasons of mathematical convenience, just as the introduction of $j = \sqrt{-1}$ often simplifies calculations in alternating current theory.

The phenomena of time dilation and length contraction can also be illustrated using the ideas developed in this section. Consider a clock at the origin of Σ which emits signals corresponding to $\Delta X_4 = 1$. These events are represented by a series of crosses along the X_4 axis as shown in *Figure 6.9*. In order to obtain the X_4' co-ordinates of these events draw a series of lines parallel to the X_1'

axis. The distance between each cross on the X_4' axis is equal to $\Delta X_4 \cos\phi$. From eqn (6.19), $\cos\phi = 1/(1 - v^2/c^2)^{\frac{1}{2}}$, hence

$$\Delta X_4' = \frac{\Delta X_4}{\sqrt{1 - v^2/c^2}}$$

that is

$$ic\,\Delta t' = \frac{ic\,\Delta t}{\sqrt{1 - v^2/c^2}}, \quad \text{or} \quad \Delta t' = \frac{\Delta t}{\sqrt{1 - v^2/c^2}}$$

This represents the phenomenon of time dilation.

If a rod of length l_0 is at rest along the X_1 axis of the inertial frame Σ, the world lines of its end points are represented by the lines AA' and BB' in *Figure 6.9*. To determine the length of the rod in Σ', the ends of the rod, which is moving with uniform velocity v relative to Σ', must be measured at the same time in Σ'. If the length of the rod relative to Σ' is measured at a time $X_4' = 0$, then its length is given by the difference between the intercepts cut by the 'world lines' AA' and BB' on the X_1' axis, that is, by DD' as shown in *Figure 6.9*. We have

$$DD' = l = l_0 \sec\phi = l_0\sqrt{1 - v^2/c^2}$$

This gives the length contraction of a moving rod. It is quite straightforward to show again that both time dilation and length contraction are reciprocal.

6.4. THE DEVELOPMENT OF THE THEORY OF SPECIAL RELATIVITY USING 4-VECTORS

In elementary textbooks a vector is defined as a quantity which has both magnitude and direction. The most convenient way of representing a vector is to use a co-ordinate system fixed in space. The directions of the axes of the co-ordinate system represent fixed directions in space. The vector can then be represented by its components along the various axes. If the components of a vector are given, then the magnitude of the vector, and its direction in space can be calculated. If the axes of the co-ordinate system are rotated in space, the vector itself must not be changed in either magnitude or direction. If this is to be the case, then the components of the vector must transform in a particular way when the axes are rotated. If the components of any quantity transform in the same way as the components of a vector, when the co-ordinate axes are rotated in space, then that quantity must be a vector. Thus it is convenient to define a vector in terms of the transformation properties of its components, when the co-ordinate axes are

rotated. A 4-vector in the $X_1X_2X_3X_4$ space will be defined as a quantity which transforms under a Lorentz transformation, in the same way as the X_1, X_2, X_3, X_4 co-ordinates of a point in the four-dimensional space. For example, **A** is defined to be a 4-vector if, under a Lorentz transformation,

$$A_1' = \gamma\left(A_1 + \mathrm{i}\frac{v}{c}A_4\right); A_2' = A_2; A_3' = A_3; A_4' = \gamma\left(A_4 - \mathrm{i}\frac{v}{c}A_1\right) \tag{6.20}$$

by analogy with the equations:

$$X_1' = \gamma\left(X_1 + \mathrm{i}\frac{v}{c}X_4\right); X_2' = X_2; X_3' = X_3; X_4' = \gamma\left(X_4 - \mathrm{i}\frac{v}{c}X_1\right) \tag{6.21}$$

Now
$$\begin{aligned} A_1'^2 + A_2'^2 + A_3'^2 + A_4'^2 &= \gamma^2\left(A_1 + \mathrm{i}\frac{v}{c}A_4\right)^2 \\ &\quad + A_2^2 + A_3^2 + \gamma^2\left(A_4 - \mathrm{i}\frac{v}{c}A_1\right)^2 \\ &= A_1^2 + A_2^2 + A_3^2 + A_4^2 \end{aligned} \tag{6.22}$$

Hence the length of a 4-vector is unchanged under rotation of axes (that is by a Lorentz transformation). If the square of the length of a 4-vector is positive it is a space-like vector, whereas if the square of its length is negative it is a *time-like* vector.

Returning to the position 4-vector, its four components can be represented by $(X_1, X_2, X_3, X_4) = (\mathbf{r}, \mathrm{i}ct)$. The three components X_1, X_2 and X_3 are the components of a vector in ordinary three-dimensional space; the fourth component is equal to a scalar times $\sqrt{-1}$. All 4-vectors have this property. If the components of another position 4-vector are $X_1 + \delta X_1$, $X_2 + \delta X_2$, $X_3 + \delta X_3$, $X_4 + \delta X_4$ then

$$(X_1' + \delta X_1') = \gamma\left[X_1 + \delta X_1 + \frac{\mathrm{i}v}{c}(X_4 + \delta X_4)\right]$$

$$X_2' + \delta X_2' = X_2 + \delta X_2$$

$$X_3' + \delta X_3' = X_3 + \delta X_3$$

$$X_4' + \delta X_4' = \gamma\left[X_4 + \delta X_4 - \mathrm{i}\frac{v}{c}(X_1 + \delta X_1)\right]$$

Subtracting the transformations for the 4-vector (X_1, X_2, X_3, X_4) given by eqn (6.21),

$$\delta X_1' = \gamma\left[\delta X_1 + \mathrm{i}\frac{v}{c}\,\delta X_4\right];\ \delta X_2' = \delta X_2;\ \delta X_3' = \delta X_3;$$

$$\delta X_4' = \gamma\left[\delta X_4 - \mathrm{i}\frac{v}{c}\,\delta X_1\right] \tag{6.23}$$

Hence the increments in a 4-vector form a 4-vector. Now the length of a 4-vector is an invariant, hence

$$\delta X_1'^2 + \delta X_2'^2 + \delta X_3'^2 + \delta X_4'^2 = \delta X_1^2 + \delta X_2^2 + \delta X_3^2 + \delta X_4^2$$

This corresponds to the relation

$$\delta s^2 = \delta x^2 + \delta y^2 + \delta z^2 - c^2\,\delta t^2 = \text{INVARIANT}$$

The above relation was derived in Section 3.12, where it was shown that the interval between two events is an invariant. It was shown in Section 3.12 that when δs^2 is negative it is possible to choose an inertial reference frame in which the events occur at the same point. The time interval between the events is then a proper time interval. Let $\delta s = \mathrm{i}c\,\delta\tau$, where $\delta\tau$ is the proper time interval.

$$\delta s^2 = (\mathrm{i}c\,\delta\tau)^2 = \delta x^2 + \delta y^2 + \delta z^2 - c^2\,\delta t^2$$

$$\frac{\delta x^2}{\delta t^2} + \frac{\delta y^2}{\delta t^2} + \frac{\delta z^2}{\delta t^2} - c^2 = \left(\frac{\mathrm{i}c\,\delta\tau}{\delta t}\right)^2 \tag{6.24}$$

If the two events, having co-ordinates X_1, X_2, X_3 and X_4 and $X_1 + \delta X_1$, $X_2 + \delta X_2$, $X_3 + \delta X_3$ and $X_4 + \delta X_4$ respectively, refer to the positions of a particle at times t and $t + \delta t$ in the inertial frame Σ, then

$$\frac{\delta x}{\delta t} = u_x;\quad \frac{\delta y}{\delta t} = u_y;\quad \frac{\delta z}{\delta t} = u_z$$

Eqn (6.24) becomes

$$u^2 - c^2 = -c^2\frac{\delta\tau^2}{\delta t^2}$$

$$\delta\tau^2 = \delta t^2(1 - u^2/c^2)$$

or

$$\delta\tau = \delta t\sqrt{1 - u^2/c^2} \tag{6.25}$$

In eqn (6.25), u is the three-dimensional velocity of the particle in the inertial frame Σ. The quantity $\delta\tau$ is an invariant since δs^2 is an invariant. If all the components of a vector are multiplied by a scalar or an invariant α, then one obtains a new vector of length α

times the original vector. Let the 4-vectors $\mathbf{X}$ and $\mathbf{X} + \delta\mathbf{X}$ refer to the positions of a particle at times t and $t + \delta t$; then

$$\delta\mathbf{X} = (\delta X_1, \delta X_2, \delta X_3, \delta X_4)$$

is a 4-vector. If this 4-vector is multiplied by the invariant $1/\delta\tau$, then one obtains a quantity which is also a 4-vector; let this be denoted by $\mathbf{U}$. One has

$$\mathbf{U} = \left(\frac{\delta X_1}{\delta\tau}, \frac{\delta X_2}{\delta\tau}, \frac{\delta X_3}{\delta\tau}, \frac{\delta X_4}{\delta\tau}\right)$$

$$= \left(\frac{\mathrm{d}x}{\sqrt{(1 - u^2/c^2)}\,\mathrm{d}t}, \frac{\mathrm{d}y}{\sqrt{(1 - u^2/c^2)}\,\mathrm{d}t}, \frac{\mathrm{d}z}{\sqrt{(1 - u^2/c^2)}\,\mathrm{d}t}, \frac{\mathrm{i}c\,\mathrm{d}t}{\sqrt{(1 - u^2/c^2)}\,\mathrm{d}t}\right)$$

But $\mathrm{d}x/\mathrm{d}t = u_x$, etc, where $\mathbf{u}$ is the ordinary three-dimensional velocity of the particle, hence

$$\mathbf{U} = \left(\frac{u_x}{\sqrt{1 - u^2/c^2}}, \frac{u_y}{\sqrt{1 - u^2/c^2}}, \frac{u_z}{\sqrt{1 - u^2/c^2}}, \frac{\mathrm{i}c}{\sqrt{1 - u^2/c^2}}\right)$$

$$= \left(\frac{\mathbf{u}}{\sqrt{1 - u^2/c^2}}, \frac{\mathrm{i}c}{\sqrt{1 - u^2/c^2}}\right) \qquad (6.26)$$

The 4-vector $\mathbf{U}$ is called the 4-vector velocity or the 4-velocity. In Σ' the 4-velocity of the particle is given by

$$\mathbf{U}' = \left(\frac{\mathbf{u}'}{\sqrt{1 - u'^2/c^2}}, \frac{\mathrm{i}c}{\sqrt{1 - u'^2/c^2}}\right) \qquad (6.27)$$

where $\mathbf{u}'$ is the three-dimensional velocity of the particle measured in Σ'. If $\mathbf{U}$ is a 4-vector it must transform according to eqns (6.20), hence

$$U_4' = \gamma\left(U_4 - \mathrm{i}\frac{v}{c}\,U_1\right)$$

$$= \gamma\left[\frac{\mathrm{i}c}{\sqrt{1 - u^2/c^2}} - \mathrm{i}\frac{v}{c}\frac{u_x}{\sqrt{1 - u^2/c^2}}\right] = \frac{\mathrm{i}c\gamma\left(1 - \dfrac{vu_x}{c^2}\right)}{\sqrt{1 - u^2/c^2}}$$

But from eqn (6.27)

$$U_4' = \frac{\mathrm{i}c}{\sqrt{1 - u'^2/c^2}}$$

Hence

$$\frac{1}{\sqrt{1 - u'^2/c^2}} = \frac{(1 - vu_x/c^2)}{\sqrt{1 - v^2/c^2}\sqrt{1 - u^2/c^2}} \tag{6.28}$$

This is the same as eqn (4.14). From eqn (6.26)

$$U_1' = \gamma\left[U_1 + \mathrm{i}\frac{v}{c}U_4\right] = \gamma\left[\frac{u_x}{\sqrt{1 - u^2/c^2}} + \mathrm{i}\frac{v}{c}\frac{\mathrm{i}c}{\sqrt{1 - u^2/c^2}}\right]$$
$$= \gamma\frac{(u_x - v)}{\sqrt{1 - u^2/c^2}}$$

But from eqn (6.27), U_1' is equal to $\dfrac{u_x'}{\sqrt{1 - u'^2/c^2}}$

Hence,

$$u_x' = \frac{\gamma(u_x - v)\sqrt{1 - u'^2/c^2}}{\sqrt{1 - u^2/c^2}}$$

Using eqn (6.28),

$$u_x' = \frac{(u_x - v)}{(1 - vu_x/c^2)}$$

This expression is the transformation for the x component of the 3-dimensional velocity of the particle.

From eqns (6.20) $U_2' = U_2$, hence:

$$\frac{u_y'}{\sqrt{1 - u'^2/c^2}} = \frac{u_y}{\sqrt{1 - u^2/c^2}}, \quad \text{or} \quad u_y' = u_y\frac{\sqrt{1 - u'^2/c^2}}{\sqrt{1 - u^2/c^2}}$$

Using eqn (6.28),

$$u_y' = \frac{u_y\sqrt{1 - v^2/c^2}}{(1 - vu_x/c^2)}$$

Similarly,

$$u_z' = \frac{u_z\sqrt{1 - v^2/c^2}}{(1 - vu_x/c^2)}$$

This completes the velocity transformations. The length of a 4-vector must be an invariant. Now

$$U^2 = U_1^2 + U_2^2 + U_3^2 + U_4^2$$
$$= \frac{u^2}{(1 - u^2/c^2)} + \frac{\mathrm{i}^2c^2}{(1 - u^2/c^2)}$$
$$= \frac{u^2 - c^2}{(1 - u^2/c^2)} = -\frac{c^2(1 - u^2/c^2)}{(1 - u^2/c^2)} = -c^2$$

According to the postulate of the constancy of the velocity of light, this must be an invariant. The 4-velocity is a time like 4-vector.

If $\mathbf{U}$ is the 4-velocity of a particle at the point x, y, z at a time t in the inertial frame Σ and if $\mathbf{U} + \delta\mathbf{U}$ is its 4-velocity at the point $x + \delta x$, $y + \delta y$, $z + \delta z$ at a time $t + \delta t$, then the increments in the components of $\mathbf{U}$ are the components of a 4-vector, that is $(\delta U_1, \delta U_2, \delta U_3, \delta U_4)$ is a 4-vector. Multiplying by the invariant $1/\delta\tau$ one obtains another 4-vector, $\mathbf{A}$, which is called the 4-vector acceleration or the 4-acceleration. After some algebra one finds

$$\mathbf{A} = \frac{\mathrm{d}\mathbf{U}}{\mathrm{d}\tau} = \left(\left\{\frac{c^2}{(c^2 - u^2)}\,\mathbf{a} + \mathbf{u}\,\frac{c^2(\mathbf{u}\cdot\mathbf{a})}{(c^2 - u^2)^2}\right\}, \frac{ic^3(\mathbf{u}\cdot\mathbf{a})}{(c^2 - u^2)^2}\right) \tag{6.29}$$

where $\mathbf{a}$ is the 3-dimensional acceleration of the particle.

In the theory of special relativity the inertial rest mass of a particle should have the same numerical value when the particle is at *rest* in all inertial frames, though when it is moving the mass of a particle depends on the velocity of the particle. Multiplying the 4-velocity by the invariant rest mass m_0 one obtains

$$\mathbf{P} = m_0\mathbf{U} = \left(\frac{m_0\mathbf{u}}{\sqrt{1 - u^2/c^2}}, \frac{im_0c}{\sqrt{1 - u^2/c^2}}\right) = (\mathbf{p}, imc) \tag{6.30}$$

$\mathbf{P}$ is called the 4-vector momentum or the 4-momentum, whilst $\mathbf{p} = m_0\mathbf{u}/(1 - u^2/c^2)^{\frac{1}{2}}$ is the 3-dimensional momentum, and $m = m_0/(1 - u^2/c^2)^{\frac{1}{2}}$ is the relativistic mass. At this stage one can proceed precisely as in Section 5.3 defining the 3-dimensional force $\mathbf{f}$ as $\mathrm{d}\mathbf{p}/\mathrm{d}t$, and defining the work done in a displacement $\mathrm{d}\mathbf{l}$ as $\mathbf{f}\cdot\mathrm{d}\mathbf{l}$. Equating the work done to the increase in kinetic energy, it was found in Section 5.3 that

$$\mathrm{d}T = \mathbf{f}\cdot\mathrm{d}\mathbf{l} \qquad \text{or} \qquad \frac{\mathrm{d}T}{\mathrm{d}t} = \mathbf{f}\cdot\frac{\mathrm{d}\mathbf{l}}{\mathrm{d}t} = \mathbf{f}\cdot\mathbf{u} \tag{6.31}$$

Integrating, as in Section 5.3, one obtains

$$T = \frac{m_0c^2}{\sqrt{1 - u^2/c^2}} - m_0c^2$$

or

$$mc^2 = T + m_0c^2 = E$$

Substituting for mc^2 in eqn (6.30),

$$\mathbf{P} = (\mathbf{p}, \mathrm{i}E/c) \tag{6.32}$$

The corresponding 4-vector in Σ' is

$$\mathbf{P}' = (\mathbf{p}', \mathrm{i}m'c) = (\mathbf{p}', \mathrm{i}E'/c) \tag{6.33}$$

The length of a 4-vector is an invariant, therefore

$$p^2 + (\mathrm{i}E/c)^2 = \text{constant}$$

or

$$p^2 - \frac{E^2}{c^2} = \text{constant}$$

When $p = 0$, $E = m_0c^2$ so that the constant is equal to $-m_0^2c^2$, so that

$$E^2 = p^2c^2 + m_0^2c^4$$

Since the length of a 4-vector is an invariant, in Σ' one has

$$p'^2 - E'^2/c^2 = -m_0^2c^2$$

giving

$$E'^2 = p'^2c^2 + m_0^2c^4$$

To obtain the transformations for the momentum of a single particle, use can be made of the fact that $(\mathbf{p}, \mathrm{i}E/c)$ is a 4-vector. For the first component, one has, from eqn (6.20)

$$P_1' = \gamma\left(P_1 + \frac{\mathrm{i}v}{c}P_4\right)$$

or

$$p_x' = \gamma\left(p_x + \frac{\mathrm{i}v}{c}\,\mathrm{i}E/c\right) = \gamma(p_x - vE/c^2)$$

From eqn (6.20),

$$P_2' = P_2 \quad \text{giving} \quad p_y' = p_y$$

Similarly

$$P_3' = P_3 \quad \text{gives} \quad p_z' = p_z$$

The relation $P_4' = \gamma\left(P_4 - \frac{\mathrm{i}v}{c}P_1\right)$ gives

$$\mathrm{i}E'/c = \gamma\left(\mathrm{i}E/c - \frac{\mathrm{i}v}{c}p_x\right)$$

or

$$E' = \gamma(E - vp_x)$$

These transformations are the same as those derived in Section 5.5.

Now from eqn (6.32) it follows that $\mathrm{d}\mathbf{P} = \left(\mathrm{d}\mathbf{p}, \mathrm{i}\frac{\mathrm{d}E}{c}\right)$ is a 4-vector. Multiplying by $1/\mathrm{d}\tau$ one obtains the Minkowski 4-dimensional force $\mathbf{F}$, which is defined by the relation

$$\mathbf{F} = \frac{\mathrm{d}\mathbf{P}}{\mathrm{d}\tau} = \left(\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}\tau}, \frac{\mathrm{i}}{c}\frac{\mathrm{d}E}{\mathrm{d}\tau}\right) = \left(\frac{\mathrm{d}\mathbf{p}}{\sqrt{(1-u^2/c^2)}\mathrm{d}t}, \frac{\mathrm{i}}{c}\frac{\mathrm{d}E}{\sqrt{(1-u^2/c^2)}\mathrm{d}t}\right)$$

If the rest mass of the particle is an invariant, using eqn (6.30)

$$\mathbf{F} = m_0 \frac{\mathrm{d}\mathbf{U}}{\mathrm{d}\tau} \tag{6.34}$$

(The rest mass is not always invariant, for example the rest mass of a rocket burning up fuel and emitting gases is reducing continually.)

Now

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(T + m_0c^2) = \frac{\mathrm{d}T}{\mathrm{d}t}$$

From eqn (6.31), $\mathrm{d}T/\mathrm{d}t = \mathbf{f} \cdot \mathbf{u}$, where $\mathbf{f}$ is the 3-dimensional force, which is defined as $\mathbf{f} = \dfrac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}$. Hence,

$$\mathbf{F} = \left(\frac{\mathbf{f}}{\sqrt{1 - u^2/c^2}}, \frac{\mathrm{i}\mathbf{f} \cdot \mathbf{u}}{c\sqrt{1 - u^2/c^2}}\right) \tag{6.35}$$

The corresponding 4-vector in Σ' is

$$\mathbf{F}' = \left(\frac{\mathbf{f}'}{\sqrt{1 - u'^2/c^2}}, \frac{\mathrm{i}\,\mathbf{f}' \cdot \mathbf{u}'}{c\sqrt{1 - u'^2/c^2}}\right) \tag{6.36}$$

Now for the first component of a 4-vector, from eqn (6.20) one has

$$F_1' = \gamma\left(F_1 + \frac{\mathrm{i}v}{c} F_4\right)$$

That is,

$$\frac{f_x'}{\sqrt{1 - u'^2/c^2}} = \gamma\left[\frac{f_x}{\sqrt{1 - u^2/c^2}} + \mathrm{i}\frac{v}{c}\frac{\mathrm{i}}{c}\frac{\mathbf{f} \cdot \mathbf{u}}{\sqrt{1 - u^2/c^2}}\right]$$

$$f_x' = \frac{\sqrt{1 - u'^2/c^2}}{\sqrt{1 - v^2/c^2}\sqrt{1 - u^2/c^2}}\left[f_x - \frac{v}{c^2}(f_xu_x + f_yu_y + f_zu_z)\right] \tag{6.37}$$

But from eqn (6.28),

$$\frac{\sqrt{1 - u'^2/c^2}}{\sqrt{1 - v^2/c^2}\sqrt{1 - u^2/c^2}} = \frac{1}{\left(1 - \dfrac{vu_x}{c^2}\right)}$$

Eqn (6.37) becomes

$$f_x' = \frac{1}{\left(1 - \dfrac{vu_x}{c^2}\right)}\left[f_x\left(1 - \frac{vu_x}{c^2}\right) - \frac{v}{c^2}(f_yu_y + f_zu_z)\right]$$

that is

$$f'_x = f_x - \frac{vu_y f_y}{(c^2 - vu_x)} - \frac{vu_z f_z}{(c^2 - vu_x)} \tag{6.38}$$

Since, from eqn (6.20), $F'_2 = F_2$:

$$\frac{f'_y}{\sqrt{1 - u'^2/c^2}} = \frac{f_y}{\sqrt{1 - u^2/c^2}}$$

Using eqn (6.28),

$$\sqrt{\frac{1 - u'^2/c^2}{1 - u^2/c^2}} = \sqrt{\frac{1 - v^2/c^2}{(1 - vu_x/c^2)^2}}$$

Hence,

$$f'_y = \frac{c^2\sqrt{1 - v^2/c^2}}{(c^2 - vu_x)} f_y \tag{6.39}$$

Similarly,

$$f'_z = \frac{c^2\sqrt{(1 - v^2/c^2)}}{(c^2 - vu_x)} f_z \tag{6.40}$$

This completes the transformations for the three-dimensional force. The use of 4-vectors is a concise alternative approach to the relativistic mechanics of a single particle, the equation of motion being given by eqn (6.34). If it is postulated that in point collisions or particle decays

$$\Sigma\, \mathbf{P}_i = \text{constant} = \Sigma\, \bar{\mathbf{P}}_i \tag{6.41}$$

where $\mathbf{P}_i$ and $\bar{\mathbf{P}}_i$ are the 4-momenta of the ith particle, before and after collision, then writing eqn (6.41) into components,

$$\Sigma p_x = \Sigma \bar{p}_x; \Sigma p_y = \Sigma \bar{p}_y; \Sigma p_z = \Sigma \bar{p}_z; \Sigma \frac{iE}{c} = \Sigma \frac{i\bar{E}}{c} \quad \text{or} \quad \Sigma E = \Sigma \bar{E}$$

Thus the law of conservation of linear momentum and the law of conservation of mass-energy are both included in eqn (6.41).

Some people prefer to solve problems in relativistic mechanics, by writing all the equations down as 4-vector equations. One can extend the theorems of vector and tensor analysis to four dimensions, and these techniques often simplify the working out of problems. For these techniques as applied to electromagnetism see Rosser[1], Chapter 10. A reader interested in the development of the theory of special relativity using 4-vector methods is referred to *Relativity, The Special Theory*, by J. L. Synge, and to Møller's *The Theory of Relativity*.

In this section the imaginary fourth component was used. The theory developed so far could have been developed, precisely as above, if we had defined a 4-vector to be a quantity having four

real components $x, y, z, m = ct$ which transform according to the equations

$$x' = \gamma\left(x - \frac{v}{c}m\right); \quad y' = y; \quad z' = z; \quad m' = \gamma\left(m - \frac{vx}{c}\right) \qquad (6.42)$$

The proper time $d\tau$ is again invariant. The full development of the theory of special relativity in terms of 4-vectors using real variables only, and its extension to the theory of general relativity, requires the use of the notation and the results of differential geometry. This requires an extensive mathematical preparation beyond the scope of the present work. The interested reader is referred to such books as the one by Panofsky and Phillips. The theory of special relativity can be extended to electromagnetism, using $X_4 = ict$ as the fourth component of the 4-vectors and it is shown by Rosser[1], Chapter 10 that ordinary Cartesian tensors, which can be developed simply from matrix theory, suffice in this case.

6.5. DE BROGLIE 'WAVES'

According to eqn (6.32) the three-dimensional momentum $\mathbf{p}$ and the energy E of a particle can be combined to form the 4-vector $(p_x, p_y, p_z, iE/c) = (\mathbf{p}, iE/c)$. Under certain conditions a photon behaves like a particle, and it was illustrated in Section 5.8.3, how a momentum $\mathbf{p} = \mathbf{n}\, h\nu/c$ can be associated with a photon of energy $E = h\nu$, where ν is the frequency, h is Planck's constant and $\mathbf{n}$ is a unit vector in the direction of motion of the photon. In the case of a photon, the 4-vector $(\mathbf{p}, iE/c)$ can be rewritten as $\left(\mathbf{n}\dfrac{h\nu}{c}, \dfrac{ih\nu}{c}\right)$. Now consider a single photon moving in the xy plane of the inertial frame Σ, and in the $x'y'$ plane of Σ', such that the direction of motion of the photon makes angles of θ and θ' with the x and x' axes of Σ and Σ' respectively. From eqn (6.20), for a 4-vector $\mathbf{A}$

$$A_1' = \gamma\left(A_1 + i\frac{v}{c}A_4\right); \quad A_2' = A_2; \quad A_4' = \gamma\left(A_4 - i\frac{v}{c}A_1\right)$$

Applying these equations to the photon 4-momentum, one obtains

$$\begin{aligned} \nu' \cos\theta' &= \gamma\nu(\cos\theta - v/c) \\ \nu' \sin\theta' &= \nu \sin\theta \\ \nu' &= \gamma\nu(1 - (v/c)\cos\theta) \end{aligned}$$

These are precisely the same as eqns (4.28), (4.29) and (4.30) respectively, which were derived in Section 4.4 when the transformation of plane waves *in vacuo* was considered. This illustrates

how the photon and wave models yield precisely the same results for the Doppler effect and aberration. It also shows that the direction of propagation and frequency of a plane wave combine to form the 4-vector $(\mathbf{n}\nu, \mathrm{i}\nu)$, where $\mathbf{n}$ is a unit vector in the direction of propagation of the plane wave, and ν is the frequency of the wave. Multiplying by h/c, one obtains the 4-vector $\left(\frac{\mathbf{n}h}{\lambda}, \frac{\mathrm{i}h\nu}{c}\right)$, where h is Planck's constant. Thus, in the case of light the momentum and energy of a photon can be combined to form the 4-vector $(\mathbf{p}, \mathrm{i}E/c)$, and the wavelength and frequency of a plane monochromatic wave can be combined to form the 4-vector $\left(\frac{\mathbf{n}h}{\lambda}, \frac{\mathrm{i}h\nu}{c}\right)$.

De Broglie suggested that it might be possible to associate 'waves' with particles of finite rest mass. If this were so, by analogy with light, one would expect the momentum and energy of the particle to combine to form the 4-vector $(\mathbf{p}, \mathrm{i}E/c)$, whilst the wavelength and frequency of the particle 'waves' might be expected to combine to form the 4-vector $\left(\frac{\mathbf{n}h}{\lambda}, \frac{\mathrm{i}h\nu}{c}\right)$, where $\mathbf{n}$ is a unit vector in the direction of propagation of the particle 'waves'. De Broglie suggested that Planck's relation $E = h\nu$ might be valid for particles, in which case the 'wave' 4-vector can be re-written as $\left(\frac{\mathbf{n}h}{\lambda}, \frac{\mathrm{i}E}{c}\right)$. Comparing with the 4-vector $(\mathbf{p}, \mathrm{i}E/c)$, one would then expect that $\mathbf{p}$ should be equal to $\mathbf{n}\, h/\lambda$, or

$$\lambda = h/p \tag{6.43}$$

This is the de Broglie relation; it has been confirmed by experiment. It would be beyond the scope of the present text to discuss the nature of de Broglie 'waves' and the extension of the theory to quantum mechanics. The interested reader is referred to a textbook on quantum mechanics. The de Broglie relation was not deduced from the theory of special relativity in the sense that it follows logically from the principle of relativity and the principle of the constancy of the velocity of light. What has been shown is that the relation $\lambda = h/p$ is consistent with the theory of special relativity. The phase velocity v_{ph} of de Broglie waves is given by

$$v_{ph} = \lambda\nu = \frac{h}{p}\frac{E}{h} = \frac{E}{p} = \frac{m_0c^2}{\sqrt{1 - u^2/c^2}} \div \frac{m_0u}{\sqrt{1 - u^2/c^2}} = \frac{c^2}{u}$$

Hence, the phase velocity of particle 'waves' exceeds the velocity of light. On the other hand, the group velocity v_g is given by

$$v_g = \frac{\mathrm{d}\nu}{\mathrm{d}\left(\frac{1}{\lambda}\right)} = \frac{\mathrm{d}(h\nu)}{\mathrm{d}\left(\frac{h}{\lambda}\right)} = \frac{\mathrm{d}E}{\mathrm{d}p}$$

$$= \frac{\mathrm{d}}{\mathrm{d}u}\left[\frac{m_0c^2}{(1 - u^2/c^2)^{\frac{1}{2}}}\right] \div \frac{\mathrm{d}}{\mathrm{d}u}\left[\frac{m_0u}{(1 - u^2/c^2)^{\frac{1}{2}}}\right]$$

$$= \frac{m_0u}{(1 - u^2/c^2)^{\frac{3}{2}}} \div \frac{m_0}{(1 - u^2/c^2)^{\frac{3}{2}}} = u$$

Hence, the group velocity of de Broglie waves is equal to the velocity of the particle.

REFERENCES

[1] ROSSER, W. G. V. *An Introduction to the Theory of Relativity*. 1964. London; Butterworths

RECOMMENDED READING

Minkowski's original article is translated in *The Principle of Relativity*, a collection of original papers published by Dover Publications Inc. Good accounts of the geometrical representation of the Lorentz transformations will be found in Born's *Einstein's Theory of Relativity*, French's *Principles of Modern Physics*, Eddington's *Space, Time and Gravitation*, and Ahoroni's *The Special Theory of Relativity*. For accounts of a simple geometrical method of representing the Lorentz transformations between two inertial reference frames only, the reader is referred to R. W. Brehme (*Amer. J. Phys*. **30** (1962) 489, **31** (1963) 517) and to F. W. Sears (*Amer. J. Phys*. **31** (1963) 269).

For the development of the theory of special relativity using 4-vectors, the reader is referred to Møller's *The Theory of Relativity*, to *Classical Electricity and Magnetism* by Panofsky and Phillips and to *Relativity, The Special Theory* by J. L. Synge.

PROBLEMS

Note: In all the calculations on 4-vectors, the reader should assume that the 4-vectors are defined acording to eqn (6.20). This is consistent with Møller's *The Theory of Relativity*.

Problem 6.1—Show that $x'^2 + y'^2 + z'^2 - c^2t'^2$ is transformed into $x^2 + y^2 + z^2 - c^2t^2$ by the transformation

$$x' = x\cosh\alpha - ct\sinh\alpha;\ y' = y;\ z' = z;$$
$$ct' = -x\sinh\alpha + ct\cosh\alpha.$$

Show that if $\alpha = \tanh^{-1} v/c = \frac{1}{2}\log_e \frac{(c + v)}{(c - v)}$, the above transformations are the same as the Lorentz transformations.

Problem 6.2—Show that $x'^2 + y'^2 + z'^2 - c^2t'^2$ is transformed into $x^2 + y^2 + z^2 - c^2t^2$ by the transformation

$$\begin{aligned} x' &= x\cos\phi + \mathrm{i}ct\sin\phi \\ y' &= y;\ \ z' = z; \\ \mathrm{i}ct' &= -x\sin\phi + \mathrm{i}ct\cos\phi \end{aligned}$$

Show that, if $\tan\phi = \mathrm{i}v/c$, the above transformations are the same as the Lorentz transformations.

Problem 6.3—Explain the use of the Minkowski diagram using x and $m = ct$ as variables. What is the significance of world lines? What are the calibration hyperbolae?

Discuss in terms of a Minkowski diagram the concepts of (*a*) length contraction; (*b*) time dilation. Use geometrical methods to calculate the formulae for length contraction and time dilation.

Problem 6.4—Explain how the Lorentz transformations can be represented as a rotation in the $X_1 = x$, $X_2 = y$, $X_3 = z$, $X_4 = \mathrm{i}ct$ space.

Discuss the phenomena of length contraction and time dilation in terms of a space–time diagram using X_1 and X_4 as variables.

Problem 6.5—Show that the length of a 4-vector defined by eqn (6.20) is an invariant. What is the square of the length of (*a*) the 4-velocity; (*b*) the 4-momentum?

Problem 6.6—In Section 3.12 an interval $\mathrm{d}s$ was defined to be a time-like (space-like) interval if $\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 - c^2\,\mathrm{d}t^2$ was negative (positive). Show that the 4-velocity and 4-momentum for a single particle are time-like vectors, and show that the 4-acceleration and the 4-force for a single particle are space-like vectors. [Comment: It is instructive to consider the 4-vectors in the reference frame in which the particle is instantaneously at rest. In this reference frame, the fourth (or time-like) component of the 4-acceleration and 4-force are both zero, since both are proportional to the 3-velocity of the particle. This shows that these 4-vectors are space-like. It is always possible to choose a reference frame in which the fourth component of a space-like vector is zero. Conversely, it is possible to find a reference frame in which the first three components of a time-like vector are all zero.]

Problem 6.7—The scalar product of two 4-vectors can be defined by the relation

$$\mathbf{A}\,.\,\mathbf{B} = A_1B_1 + A_2B_2 + A_3B_3 + A_4B_4$$

Show that the scalar product, so defined, is an invariant, that is, show $\mathbf{A}'\,.\,\mathbf{B}' = \mathbf{A}\,.\,\mathbf{B}$.

Problem 6.8—Show that the 4-velocity and the 4-acceleration of a particle are orthogonal, that is, show that $\mathbf{U}\,.\,\mathbf{A}$ is zero. [Hint: Use direct substitution, or start from the fact that $\mathbf{U}\,.\,\mathbf{U} = -c^2$ and differentiate with respect to τ giving $\mathbf{U}\,.\,\dfrac{\mathrm{d}\mathbf{U}}{\mathrm{d}\tau} = 0$]

Problem 6.9—The Minkowski 4-force on a particle is defined by the relation $\mathbf{F} = \dfrac{\mathrm{d}\mathbf{P}}{\mathrm{d}\tau}$.

Show that if the rest mass of the particle is a constant, then $\mathbf{F}\cdot\mathbf{U} = 0$, whereas if the rest mass is not constant $\mathbf{F}\cdot\mathbf{U} = -c^2\dfrac{\mathrm{d}m_0}{\mathrm{d}\tau}$ [Hint: $\mathbf{F} = \dfrac{\mathrm{d}}{\mathrm{d}\tau}(m_0\mathbf{U}) = \mathbf{U}\dfrac{\mathrm{d}m_0}{\mathrm{d}\tau} + m_0\dfrac{\mathrm{d}\mathbf{U}}{\mathrm{d}\tau}$. Form $\mathbf{F}\cdot\mathbf{U}$ and remember $\mathbf{U}\cdot\mathbf{U} = -c^2$].

Problem 6.10—Show that if $\mathbf{r}(t)$ is the position vector of a moving point in a co-ordinate system Σ, then $\mathrm{d}\tau = (\mathrm{d}t^2 - \mathrm{d}r^2/c^2)^{\frac{1}{2}}$ is an invariant for all inertial frames Σ in uniform relative motion, and deduce that the Minkowski velocity and acceleration are, respectively, $(\gamma\mathbf{u},\ \mathrm{i}\gamma c)$ and $(\gamma^2(\mathbf{a} + \mathbf{u}\theta),\ \mathrm{i}c\gamma^2\theta)$, where $\mathbf{u}$ and $\mathbf{a}$ are the Newtonian 3-velocity and 3-acceleration, $\gamma = (1 - u^2/c^2)^{-\frac{1}{2}}$ and $\theta = \gamma^2\mathbf{u}\cdot\mathbf{a}/c^2$.

Hence, or otherwise, show that if $\mathbf{a}_0$ is the acceleration in the rest frame (in which the velocity is zero instantaneously) the corresponding acceleration $\mathbf{a}$ in the frame in which the particle has velocity $\mathbf{u}$ is given by

$$\mathbf{a} = \frac{1}{\gamma^2}\left\{\mathbf{a}_0 - \left(\frac{1-\gamma}{u^2} + \frac{\gamma}{c^2}\right)(\mathbf{u}\cdot\mathbf{a}_0)\mathbf{u}\right\} \qquad \text{(Exeter 1961)}$$

Problem 6.11—A particle of rest mass m_0 describes the parabolic trajectory $x = ut,\ y = \frac{1}{2}at^2,\ z = 0$ in the inertial frame Σ. Find the four components in the Σ-system of the Minkowski 4-force vector, and the corresponding Newtonian force on the particle. [Hint: Use eqn (6.35). As a check for the Minkowski 4-force $\mathbf{F}$, we have $\mathbf{F} = m_0\mathbf{A}$, where $\mathbf{A}$ is the 4-acceleration. Comparing with *Problem 6.10*, one can write $\mathbf{F} = (m_0\gamma^2u\theta,\ m_0\gamma^2(a + at\theta),\ 0,\ \mathrm{i}m_0c\gamma^2\theta)$ where $\gamma^2 = 1/(1 - u^2/c^2 - a^2t^2/c^2)$; $\theta = \gamma^2a^2t/c^2$.]

Problem 6.12—Define the Minkowski force $(\mathbf{G}, \mathrm{i}H)$ on a particle, and show that

$$\mathbf{G} = \mathbf{f}_0 - (1-\gamma)(\mathbf{v}\cdot\mathbf{f}_0)\mathbf{v}/v^2; \quad H = \gamma(\mathbf{v}\cdot\mathbf{f}_0)/c$$

where $\mathbf{v}$ is the velocity of the particle in the Σ frame, $\mathbf{f}_0$ is the Newtonian force in the rest frame and $\gamma = (1 - v^2/c^2)^{-\frac{1}{2}}$.

Deduce that in rectilinear motion, the Newtonian force in the Σ frame (G/γ) is independent of $\mathbf{v}$.

A particle of rest mass m_0 is acted on by a constant force m_0f along the axis Ox and is at rest at O at time $t = 0$. Find the distance it will travel in time t, and show that its momentum when it has travelled a distance x is $m_0(2xf + x^2f^2/c^2)^{\frac{1}{2}}$. (Exeter 1961).

Problem 6.13—Show that the de Broglie wavelength associated with a particle of rest mass m_0 moving with velocity u is,

$$\lambda = h(1 - u^2/c^2)^{\frac{1}{2}}/m_0u$$

Problem 6.14—An electron is accelerated from rest through a potential difference of V volts. Show that its de Broglie wavelength expressed in ångströms is

$$\lambda = \frac{12{\cdot}27}{V^{\frac{1}{2}}}\left(\frac{Ve}{2m_0c^2} + 1\right)^{-\frac{1}{2}}$$

Problem 6.15—In Chapters 3 and 6 continuous world lines were used on space time diagrams to represent the motions of particles. According to

the uncertainty principle the positions of atomic particles are generally indeterminate and their precise positions unknown. How do you reconcile these viewpoints? (Comment: According to quantum mechanics the precise position of a particle is only known at the world points where its position is determined by experiment. Between these world points the position of the particle is indeterminate. However, the indeterminate behaviour between known world points is not inconsistent with special relativity, provided the wave equation used to describe the system satisfies the principle of relativity, when the co-ordinates and time are changed according to the Lorentz transformations. This is true of Dirac's equation for an electron.)

7

RELATIVISTIC ELECTROMAGNETISM

7.1. INTRODUCTION

When students are taught electromagnetism for the first time, it is inevitable that the individual laws such as Coulomb's law, Ampère's circuital law and Faraday's law of electromagnetic induction are introduced separately. As a result many students tend to think of these laws as describing completely independent phenomena, and very few are aware of even the relative orders of magnitude of the electric and magnetic forces between moving charges, let alone the intimate connection between the electric and magnetic forces between moving charges.

In Section 7.2 the ratio of the electric to the magnetic forces in a simple case will be calculated using formulae familiar to most advanced students at school. The example will then be discussed taking Coulomb's law and the transformations of the theory of special relativity as axiomatic. This will illustrate the unity of electromagnetism in a vivid way.

It was illustrated in Chapter 2 how the theory of special relativity evolved from optics and classical electromagnetism. In the theory of special relativity, Einstein extended the principle of relativity to optics and electromagnetism. For the benefit of readers familiar with Maxwell's equations, it will be shown, in outline, in Section 7.3 that Maxwell's equations do obey the principle of relativity, if the co-ordinates and time are transformed according to the Lorentz transformations. This Section can be omitted, as the results are not used elsewhere. A reader interested in a more comprehensive discussion of relativistic electromagnetism is referred to Rosser[1], Chapters 7, 8, 9 and 10 and to Rosser[2].

7.2. FORCES BETWEEN TWO PARALLEL CONVECTION CURRENTS

Consider two infinitely long, straight, thin, uniformly charged, non-conducting wires *in vacuo* lying in the xy plane and moving in the positive x direction with uniform velocity v relative to an inertial frame Σ as shown in *Figure 7.1(b)*. Let λ be the electric charge in coulombs per metre length of the wires, measured relative to Σ.

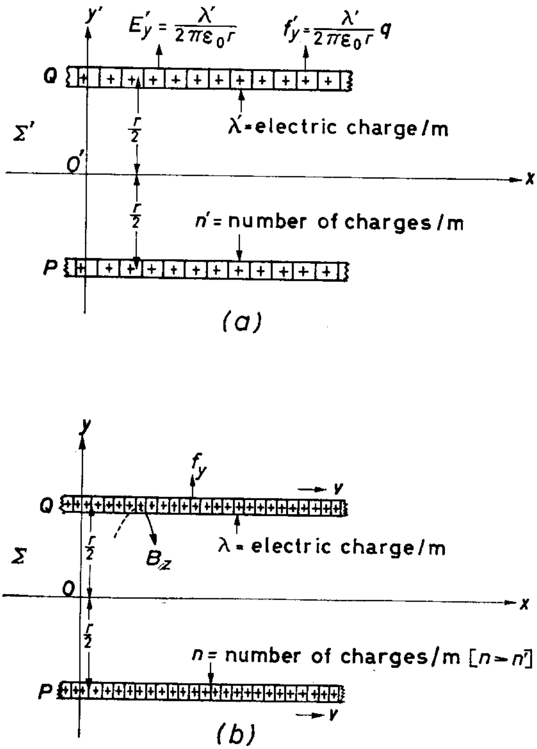

Figure 7.1. The electric and magnetic forces between two convection currents. (a) The charge distributions are at rest in Σ'; (b) the inertial frame Σ moves with a velocity $-v$ relative to Σ'. In Σ the charges move with velocity v

As the wires are moving relative to Σ, the charge distributions give rise to convection currents of magnitude λv amperes respectively.

Application of Gauss' law and Ampère's circuital law gives the electric and magnetic fields at a distance r from the wire P.

$$E_y = \frac{\lambda}{2\pi\varepsilon_0 r} \tag{7.1}$$

and

$$B_z = \frac{\mu_0 \mathrm{i}}{2\pi r} = \frac{\mu_0 \lambda v}{2\pi r} \tag{7.2}$$

The electric field gives rise to a repulsion in the y direction of $\lambda^2/2\pi\varepsilon_0 r$ N/m length of the wire Q. The magnetic field gives rise

to an attraction of $\mu_0 i^2/2\pi r$ N/m, which is equal to $\mu_0\lambda^2 v^2/2\pi r$. The resultant force on the wire Q per metre length is given by

$$\begin{aligned} f_y &= f_{\text{elec}} + f_{\text{mag}} \\ &= \frac{\lambda^2}{2\pi\varepsilon_0 r} - \frac{\mu_0\lambda^2 v^2}{2\pi r} \\ &= \frac{\lambda^2}{2\pi\varepsilon_0 r}(1 - \mu_0\varepsilon_0 v^2) \end{aligned}$$

Numerical substitution of $8{\cdot}85 \times 10^{-12}$ for ε_0 and $4\pi \times 10^{-7}$ for μ_0 shows that $\mu_0\varepsilon_0$ is equal to $1/c^2$, where c is the velocity of light. Hence,

$$f_y = \frac{\lambda^2}{2\pi\varepsilon_0 r}(1 - v^2/c^2) \tag{7.3}$$

In this simple case the ratio of the magnetic force to the electric force is v^2/c^2, where v is the velocity of the charges relative to the inertial frame of reference chosen.

In a typical case, say a current of 1 A flowing in a copper wire of cross-sectional area 1 sq. mm, the average drift velocity of the electrons is $\sim 10^{-4}$ m/sec. If $\bar{v}$ is the mean drift velocity of electrons the current is $na\bar{v}q$, where n is the number of charge carriers per unit volume, a is the area of cross-section and q is the charge on each carrier. Assuming that in copper $n \sim 8{\cdot}5 \times 10^{28}$ free electrons/m^3, and using $q = 1{\cdot}6 \times 10^{-19}$ coulombs one finds $\bar{v} \sim 7{\cdot}3 \times 10^{-5}$ m/sec. Hence the ratio of the average magnetic forces between the moving electrons in such a current to the electric forces between them is $\sim 10^{-25}$. However, in a copper conductor there is no resultant electric charge as the charges on the moving electrons are compensated by the positively charged copper ions, which are virtually at rest in a stationary conductor. For an electrically neutral conductor the electric forces on the moving electrons is compensated by the electric force on the positive ions, and the resultant force on the conductor is due to the relatively small magnetic forces between the moving charges.

The above example will now be considered from the viewpoint of the theory of special relativity. According to this theory, the laws of electromagnetism should have the same mathematical form in all inertial frames of reference. Consider the inertial frame Σ' moving with uniform velocity v relative to Σ along the positive x axis. In this reference frame the wires and the electric charge distributions are at rest as shown in *Figure 7.1*(*a*). Let the electrostatic charge distribution in Σ' be measured to be λ' coulomb/metre

and let λ' be made up of n' discrete charges per metre of magnitude q coulombs each. In Σ' Maxwell's equations reduce to Coulomb's law of force between electrostatic charges. At the position of the wire Q the electric and magnetic fields due to the wire P are given by

$$E_y' = \frac{\lambda'}{2\pi\varepsilon_0 r} \quad \text{and} \quad \mathbf{B}' = 0 \tag{7.4}$$

The force on one of the charges on the wire Q is equal to

$$f_x' = 0; \;\; f_y' = \frac{\lambda' q}{2\pi\varepsilon_0 r}; \;\; f_z' = 0 \tag{7.5}$$

It will now be *assumed* that the transformation formulae derived in Sections 5.5 and 6.4 for the force acting on a single particle are correct. From eqns (5.77), (5.78), (5.79), when $\mathbf{u}' = 0$,

$$f_x = f_x' \tag{7.6}$$

$$f_y = \sqrt{(1 - v^2/c^2)}f_y' \tag{7.7}$$

$$f_z = \sqrt{(1 - v^2/c^2)}f_z' \tag{7.8}$$

Substitution from eqns (7.5) into eqns (7.6), (7.7) and (7.8) gives

$$f_x = f_z = 0$$

$$f_y = \sqrt{1 - v^2/c^2}\,\frac{\lambda'}{2\pi\varepsilon_0 r}\,q \tag{7.9}$$

The value of the charge q is assumed to be invariant under Lorentz transformation. We have omitted the transformation of time from Σ' to Σ, namely

$$t = \frac{t' + vx'/c^2}{\sqrt{1 - v^2/c^2}}$$

If the forces are taken to be simultaneous for all x' in Σ', they are not simultaneous for all x in Σ; but we are assuming steady conditions independent of time. The length r is unchanged under Lorentz transformation, as it is measured in the y direction.

Since the wires are moving in the inertial frame Σ, a length of wire measured to be l_0 when the wires are at rest in Σ', is reduced to $l_0(1 - v^2/c^2)^{\frac{1}{2}}$ when measured in Σ. Hence the number of charges per unit length measured in Σ is greater than in Σ', as illustrated in *Figures 7.1*(*a*) and (*b*). The number of charges per unit length in Σ is given by the equation

$$n = \frac{n'}{\sqrt{1 - v^2/c^2}} \tag{7.10}$$

Hence

$$\lambda = nq = \frac{n'}{\sqrt{1 - v^2/c^2}}\, q = \frac{\lambda'}{\sqrt{1 - v^2/c^2}} \tag{7.11}$$

The force/metre on the wire Q, measured in Σ, is equal to the number of charges per metre length, measured in Σ, times the force on each charge, that is using eqn (7.9)

$$nf_v = n\sqrt{1 - v^2/c^2}\,\frac{\lambda'}{2\pi\varepsilon_0 r}\, q$$

$$= \frac{\lambda'\lambda}{2\pi\varepsilon_0 r}\sqrt{1 - v^2/c^2} \text{ since } nq = \lambda$$

Substituting for λ' from eqn (7.11), one finds that the force/metre length of Q measured in Σ is equal to

$$\frac{\lambda^2}{2\pi\varepsilon_0 r}\,(1 - v^2/c^2)$$

This is in agreement with eqn (7.3).

This example illustrates how the magnetic forces produced by currents can be calculated in some cases from Coulomb's law for the forces between electrostatic charges, if the transformation formulae for force given by the theory of special relativity and the principle of constant charge are taken as axiomatic. The magnetic forces can be interpreted as relativistic effects and represent those parts of the transformed forces which depend on the velocity of the test charge relative to the observer. For a full discussion of this approach the reader is referred to Rosser[2].

7.3. RELATIVISTIC INVARIANCE OF MAXWELL'S EQUATIONS

7.3.1. *Introduction*

The macroscopic electric and magnetic fields, both in empty space and inside *stationary* material bodies, are adequtely described by Maxwell's equations, which in an inertial frame Σ', in which the materials are at rest, take the form

$$\text{curl}'\,\mathbf{E}' = -\frac{\partial \mathbf{B}'}{\partial t'} \tag{7.12}$$

$$\text{curl}'\,\mathbf{H}' = \frac{\partial \mathbf{D}'}{\partial t'} + \mathbf{J}' \tag{7.13}$$

$$\text{div}'\,\mathbf{D}' = \rho' \tag{7.14}$$

$$\text{div}'\,\mathbf{B}' = 0 \tag{7.15}$$

In eqns (7.12)–(7.15), $\mathbf{E}'$ is the electric intensity, $\mathbf{D}'$ the electric displacement, $\mathbf{B}'$ the magnetic induction, $\mathbf{H}'$ the magnetizing force, ρ' the true macroscopic charge density and $\mathbf{J}'$ the true macroscopic conduction current density. All the quantities appearing in eqns (7.12), (7.13), (7.14) and (7.15) apply to the same point x', y', z' at a time t' in the inertial frame Σ'. All the differential coefficients in the equations are with respect to x', y', z' or t', for example, eqn (7.15) is

$$\frac{\partial B_x'}{\partial x'} + \frac{\partial B_y'}{\partial y'} + \frac{\partial B_z'}{\partial z'} = 0$$

One of the postulates of the theory of special relativity is that the laws of physics have the same mathematical form in all inertial frames of reference. If Maxwell's equations are correct and obey the principle of relativity, then in the inertial frame Σ moving with velocity v in the negative Ox' direction relative to Σ', one should have

$$\operatorname{curl} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{7.16}$$

$$\operatorname{curl} \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J} \tag{7.17}$$

$$\operatorname{div} \mathbf{D} = \rho \tag{7.18}$$

$$\operatorname{div} \mathbf{B} = 0 \tag{7.19}$$

These equations should hold for a point x, y, z at a time t in Σ, where x, y, z and t are related to x', y', z' and t' by the Lorentz transformations. It will be assumed throughout Section 7.3 that Σ' moves with uniform velocity v relative to Σ along their common x axis, so that v and $\gamma = (1 - v^2/c^2)^{-\frac{1}{2}}$ are constants.

7.3.2. *Transformation of $\partial/\partial x$, $\partial/\partial y$, $\partial/\partial z$ and $\partial/\partial t$*

Consider a function F which is a function of x', y', z' and t' in Σ'. The total differential of F is

$$\mathrm{d}F = \frac{\partial F}{\partial x'}\,\mathrm{d}x' + \frac{\partial F}{\partial y'}\,\mathrm{d}y' + \frac{\partial F}{\partial z'}\,\mathrm{d}z' + \frac{\partial F}{\partial t'}\,\mathrm{d}t' \tag{7.20}$$

Now for a given event, x', y', z' and t' are all functions of x, y, z and t. The total differential of x' can be expressed as

$$\mathrm{d}x' = \frac{\partial x'}{\partial x}\,\mathrm{d}x + \frac{\partial x'}{\partial y}\,\mathrm{d}y + \frac{\partial x'}{\partial z}\,\mathrm{d}z + \frac{\partial x'}{\partial t}\,\mathrm{d}t \tag{7.21}$$

According to the Lorentz transformations

$$x' = \gamma(x - vt); \quad y' = y; \quad z' = z$$
$$t' = \gamma(t - vx/c^2); \quad \gamma = (1 - v^2/c^2)^{-\frac{1}{2}}$$

Remembering that v and γ are constants, we have

$$\frac{\partial x'}{\partial x} = \gamma; \quad \frac{\partial x'}{\partial y} = 0; \quad \frac{\partial x'}{\partial z} = 0; \quad \frac{\partial x'}{\partial t} = -\gamma v$$

Substituting in eqn (7.21)

$$\mathrm{d}x' = \gamma\,\mathrm{d}x - \gamma v\,\mathrm{d}t \tag{7.22}$$

Similarly,

$$\mathrm{d}y' = \mathrm{d}y \tag{7.23}$$

$$\mathrm{d}z' = \mathrm{d}z \tag{7.24}$$

$$\begin{aligned}\mathrm{d}t' &= \frac{\partial t'}{\partial x}\,\mathrm{d}x + \frac{\partial t'}{\partial y}\,\mathrm{d}y + \frac{\partial t'}{\partial z}\,\mathrm{d}z + \frac{\partial t'}{\partial t}\,\mathrm{d}t \\ &= -\frac{\gamma v}{c^2}\,\mathrm{d}x + \gamma\,\mathrm{d}t \end{aligned} \tag{7.25}$$

Substituting for $\mathrm{d}x'$, $\mathrm{d}y'$, $\mathrm{d}z'$ and $\mathrm{d}t'$ from eqns (7.22), (7.23), (7.24) and (7.25) respectively into eqn (7.20), we have

$$\mathrm{d}F = \frac{\partial F}{\partial x'}(\gamma\,\mathrm{d}x - \gamma v\,\mathrm{d}t) + \frac{\partial F}{\partial y'}\,\mathrm{d}y + \frac{\partial F}{\partial z'}\,\mathrm{d}z + \frac{\partial F}{\partial t'}\left(\gamma\,\mathrm{d}t - \frac{\gamma v}{c^2}\,\mathrm{d}x\right)$$

Rearranging,

$$\begin{aligned}\mathrm{d}F = \left(\gamma\frac{\partial F}{\partial x'} - \frac{\gamma v}{c^2}\frac{\partial F}{\partial t'}\right)\mathrm{d}x + \frac{\partial F}{\partial y'}\,\mathrm{d}y + \frac{\partial F}{\partial z'}\,\mathrm{d}z \\ + \left(\gamma\frac{\partial F}{\partial t'} - \gamma v\frac{\partial F}{\partial x'}\right)\mathrm{d}t\end{aligned} \tag{7.26}$$

But, if F is a function of x, y, z and t, the total differential of F can be written as

$$\mathrm{d}F = \frac{\partial F}{\partial x}\,\mathrm{d}x + \frac{\partial F}{\partial y}\,\mathrm{d}y + \frac{\partial F}{\partial z}\,\mathrm{d}z + \frac{\partial F}{\partial t}\,\mathrm{d}t \tag{7.27}$$

Comparing the coefficients of $\mathrm{d}x$, $\mathrm{d}y$, $\mathrm{d}z$ and $\mathrm{d}t$ in eqns (7.26) and (7.27) we obtain

$$\frac{\partial}{\partial x} = \gamma\left(\frac{\partial}{\partial x'} - \frac{v}{c^2}\frac{\partial}{\partial t'}\right) \tag{7.28}$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial y'} \tag{7.29}$$

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial z'} \tag{7.30}$$

$$\frac{\partial}{\partial t} = \gamma\left(\frac{\partial}{\partial t'} - v\frac{\partial}{\partial x'}\right) \tag{7.31}$$

7.3.3. *Transformations for* **E** *and* **B**

The y component of eqn (7.16) is

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\frac{\partial B_y}{\partial t}$$

Substituting for $\partial/\partial z$, $\partial/\partial x$ and $\partial/\partial t$ from eqns (7.30), (7.28) and (7.31) respectively,

$$\frac{\partial E_x}{\partial z'} - \gamma\left(\frac{\partial E_z}{\partial x'} - \frac{v}{c^2}\frac{\partial E_z}{\partial t'}\right) = -\gamma\left(\frac{\partial B_y}{\partial t'} - v\frac{\partial B_y}{\partial x'}\right)$$

Rearranging,

$$\frac{\partial E_x}{\partial z'} - \frac{\partial}{\partial x'}\gamma(E_z + vB_y) = -\frac{\partial}{\partial t'}\gamma\left(B_y + \frac{v}{c^2}E_z\right) \tag{7.32}$$

If Maxwell's equations are to be Lorentz covariant, that is invariant in mathematical form in all inertial frames of reference, then in Σ' one must have

$$\frac{\partial E'_x}{\partial z'} - \frac{\partial E'_z}{\partial x'} = -\frac{\partial B'_y}{\partial t'} \tag{7.33}$$

Eqns (7.32) and (7.33) have the same mathematical form showing that the y-component of eqn (7.16) is Lorentz covariant. In fact, if one puts

$$E'_x = E_x$$
$$E'_z = \gamma(E_z + vB_y)$$
$$B'_y = \gamma\left(B_y + \frac{v}{c^2}E_z\right)$$

then eqns (7.32) and (7.33) are exactly the same.

It is left as an exercise for the reader to show that the x and z components of eqn (7.16) transform into the x' and z' components of eqn (7.12) respectively, and to show that eqn (7.19) transforms into eqn (7.15), if **E** and **B** transform according to the following equations (Reference: Rosser[1], Section 8.2):

$$\left.\begin{aligned} E'_x &= E_x & E_x &= E'_x \\ E'_y &= \gamma(E_y - vB_z) & E_y &= \gamma(E'_y + vB'_z) \\ E'_z &= \gamma(E_z + vB_y) & E_z &= \gamma(E'_z - vB'_y) \end{aligned}\right\} \tag{7.34}$$

$$\left.\begin{aligned} B'_x &= B_x & B_x &= B'_x \\ B'_y &= \gamma\left(B_y + \frac{v}{c^2}E_z\right) & B_y &= \gamma\left(B'_y - \frac{v}{c^2}E'_z\right) \\ B'_z &= \gamma\left(B_z - \frac{v}{c^2}E_y\right) & B_z &= \gamma\left(B'_z + \frac{v}{c^2}E'_y\right) \end{aligned}\right\} \tag{7.35}$$

These transformations relate the numerical values of the electric and magnetic fields at a point x, y, z at a time t in Σ with the values of the electric and magnetic fields at a point x', y', z' at a time t' in Σ', where x, y, z and t are related to x', y', z' and t' by the Lorentz transformations. For a full account of the applications of eqns (7.34) and (7.35) the reader is referred to Rosser[1], Chapter 8.

7.3.4. *Transformation of Charge and Current Densities*

We shall start with a simplified model and consider a 'uniform' electric charge distribution of volume V in the inertial frame Σ, consisting of n discrete charges per cubic metre, of magnitude q coulombs each. It will be assumed that all the charges have the same velocity $\mathbf{u}$ relative to the inertial frame Σ. For this simplified case the charge density is

$$\rho = nq \tag{7.36}$$

The current density, which is the current crossing unit area normal to the direction of current flow, is given by

$$\mathbf{J} = nq\mathbf{u} \tag{7.37}$$

or in components, $J_x = nqu_x$; $J_y = nqu_y$; $J_z = nqu_z$.

Let the same charge distribution have a volume V' when measured relative to an inertial frame Σ' which is moving with uniform velocity v relative to Σ, and let it consist of n' discrete charges/m^3 moving with velocity $\mathbf{u}'$ relative to Σ'. One then has

$$\rho' = n'q \tag{7.38}$$

$$\mathbf{J}' = n'q\mathbf{u}' \tag{7.39}$$

The principle of the invariance of total electric charge will be taken as axiomatic in the present section. According to this principle, the total charge on a body is independent of the velocity of the body, and the total charge has the same numerical value in all inertial reference frames. In Section 5.4, it was assumed that the magnitude of a moving charge in the expression for the Lorentz force was independent of the velocity of the charge. It was shown that this assumption was in agreement with experiments. Hence, the same value for q is used in eqns (7.36), (7.37), (7.38) and (7.39).

Consider an inertial frame Σ^0 in which the charge distribution is at rest; Σ^0 moves with velocity $-\mathbf{u}$ relative to Σ. Let V_0 be the proper volume of the charge distribution in Σ^0. Let n_0 be the number of charges/m^3 measured in Σ^0, such that the total number of charges is equal to n_0V_0. Due to length contraction, the volume

of the charge distribution in Σ should be measured to be

$$V_0(1 - u^2/c^2)^{\frac{1}{2}}$$

The total number of charges measured in Σ is equal to nV, that is, $nV_0(1 - u^2/c^2)^{\frac{1}{2}}$. But the total *number* of charges is an invariant, since it is a pure number. Hence,

$$nV = nV_0\sqrt{(1 - u^2/c^2)} = n_0V_0$$

or

$$n = \frac{n_0}{\sqrt{(1 - u^2/c^2)}}$$

Similarly in Σ',

$$n' = \frac{n_0}{\sqrt{(1 - u'^2/c^2)}}$$

so that

$$n' = n\sqrt{\frac{(1 - u^2/c^2)}{(1 - u'^2/c^2)}} \tag{7.40}$$

From eqn (4.14),

$$\sqrt{\frac{(1 - u^2/c^2)}{(1 - u'^2/c^2)}} = \frac{(1 - vu_x/c^2)}{\sqrt{(1 - v^2/c^2)}} \tag{4.14}$$

Substituting in eqn (7.40), one obtains

$$n' = \frac{n(1 - vu_x/c^2)}{\sqrt{(1 - v^2/c^2)}} = \gamma n(1 - vu_x/c^2) \tag{7.41}$$

Multiplying by q,

$$qn' = \gamma\left(qn - \frac{vqnu_x}{c^2}\right)$$

Substituting from eqns (7.38), (7.36) and (7.37),

$$\rho' = \gamma\left(\rho - \frac{vJ_x}{c^2}\right)$$

Now from eqn (7.39),

$$J'_x = qu'_xn'$$

But from the velocity transformations, one has

$$u'_x = (u_x - v)/(1 - vu_x/c^2)$$

Substituting for u'_x, and for n' from eqn (7.41),

$$J'_x = \frac{q(u_x - v)}{(1 - vu_x/c^2)}\, n\gamma(1 - vu_x/c^2)$$

Substituting from eqns (7.39), (7.36) and (7.37)

$$J'_x = \gamma(J_x - v\rho)$$

Similarly, from eqn (7.39)

$$J'_y = qu'_y n'$$

From the velocity transformations,

$$u'_y = \frac{u_y\sqrt{(1 - v^2/c^2)}}{(1 - vu_x/c^2)}$$

Substituting for u'_y and n',

$$J'_y = qu_y \frac{\sqrt{(1 - v^2/c^2)}}{(1 - vu_x/c^2)} n\gamma(1 - vu_x/c^2) = J_y$$

Similarly

$$J'_z = J_z$$

Collecting the transformations,

$$\begin{aligned} J'_x &= \gamma(J_x - v\rho) & J_x &= \gamma(J'_x + v\rho') \\ J'_y &= J_y & J_y &= J'_y \\ J'_z &= J_z & J_z &= J'_z \\ \rho' &= \gamma\left(\rho - \frac{vJ_x}{c^2}\right) & \rho &= \gamma\left(\rho' + \frac{vJ'_x}{c^2}\right) \end{aligned} \tag{7.42}$$

It is left as an exercise for the reader to extend the treatment to systems containing both positive and negative charges and to show that eqns (7.42) are valid in the general case. (Reference: Rosser[1], Section 8.4.)

7.3.5. *The Transformations for* **D** *and* **H**

Consider the y component of the equation

$$\text{curl}\,\mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \tag{7.17}$$

that is

$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = J_y + \frac{\partial D_y}{\partial t}$$

Substituting for $\partial/\partial z$, $\partial/\partial x$ and $\partial/\partial t$ from eqns (7.30), (7.28) and (7.31) and for J_y from eqn (7.42),

$$\frac{\partial H_x}{\partial z'} - \gamma\left(\frac{\partial}{\partial x'} - \frac{v}{c^2}\frac{\partial}{\partial t'}\right)H_z = J'_y + \gamma\left(\frac{\partial}{\partial t'} - v\frac{\partial}{\partial x'}\right)D_y$$

or

$$\frac{\partial H_x}{\partial z'} - \frac{\partial}{\partial x'}\gamma(H_z - vD_y) = J'_y + \frac{\partial}{\partial t'}\gamma\left(D_y - \frac{v}{c^2}H_z\right) \quad (7.43)$$

If Maxwell's equations were valid in Σ', one would have

$$\frac{\partial H'_x}{\partial z'} - \frac{\partial H'_z}{\partial x'} = J'_y + \frac{\partial D'_y}{\partial t'} \quad (7.44)$$

Eqns (7.43) and (7.44) have the same mathematical form, and if one puts

$$H'_x = H_x;\ H'_z = \gamma(H_z - vD_y);\ D'_y = \gamma\left(D_y - \frac{v}{c^2}H_z\right)$$

then they are exactly the same.

It is left as an exercise for the reader to show by similar methods that the x and z components of eqn (7.17) transform into the x' and z' components of eqn (7.13) and that eqn (7.18) transforms into eqn (7.14), if $\mathbf{J}$ and ρ are transformed using eqns (7.42), and if **H** and **D** satisfy the transformations (Reference: Rosser[1], Section 8.5):

$$\left.\begin{aligned} D'_x &= D_x & D_x &= D'_x \\ D'_y &= \gamma\left(D_y - \frac{v}{c^2}H_z\right) & D_y &= \gamma\left(D'_y + \frac{v}{c^2}H'_z\right) \\ D'_z &= \gamma\left(D_z + \frac{v}{c^2}H_y\right) & D_z &= \gamma\left(D'_z - \frac{v}{c^2}H'_y\right) \end{aligned}\right\} \quad (7.45)$$

$$\left.\begin{aligned} H'_x &= H_x & H_x &= H'_x \\ H'_y &= \gamma(H_y + vD_z) & H_y &= \gamma(H'_y - vD'_z) \\ H'_z &= \gamma(H_z - vD_y) & H_z &= \gamma(H'_z + vD'_y) \end{aligned}\right\} \quad (7.46)$$

It has been shown that if Maxwell's equations are valid in Σ, then if the co-ordinates and time are transformed using the Lorentz transformations, and if one takes the principle of constant charge as axiomatic, then the transformed equations have the same mathematical form as Maxwell's equations would have if they were valid in Σ'. This is true whether there are material bodies present at the point or not. Thus Maxwell's equations satisfy the principle of relativity when the co-ordinates and time are transformed according to the Lorentz transformations.

Only the briefest insight into the relativistic invariance of Maxwell's equations has been given. For a more comprehensive

account of relativistic electromagnetism, including an account of the applications of the transformations, the reader is referred to Rosser[1], Chapters 7, 8, 9 and 10 and to Rosser[2].

REFERENCES

[1] Rosser, W. G. V. *An Introduction to the Theory of Relativity*. 1964. London; Butterworths.

[2] Rosser, W. G. V. *Electromagnetism via Relativity, An Alternative Approach to Maxwell's Equations*, In Press, London; Butterworths.

PROBLEMS

Problem 7.1—Starting from Maxwell's equations, derive the formulae for the transformation of the electric and magnetic field vectors **E** and **B** from one inertial reference frame to another moving with uniform velocity v relative to the first. Hence, show that the electric field at a point P in empty space due to a charge of magnitude q, at O moving with uniform velocity $\mathbf{v}$ is given by

$$\mathbf{E} = \frac{q\mathbf{r}}{4\pi\varepsilon_0 r^3}(1-\beta^2)(1-\beta^2\sin^2\theta)^{-\frac{3}{2}}, \beta = v/c$$

where $\mathbf{r}$ is a vector from O to P and θ is the angle between $\mathbf{r}$ and $\mathbf{v}$. Show that the magnetic field at P is given by

$$\mathbf{B} = \mathbf{v} \times \mathbf{E}/c^2$$

Problem 7.2—A charge of magnitude q coulombs is moving with uniform velocity $\mathbf{u}$ in the inertial reference frame Σ. In the inertial frame Σ' moving with velocity $\mathbf{u}$, relative to Σ, the charge q is at rest, and the force on it is $q\mathbf{E}'$. Use the force transformations and the transformations for **E** and **B** to show that the force on the particle in Σ is $q\mathbf{E} + q\mathbf{u} \times \mathbf{B}$.

Problem 7.3—Show by direct substitution that the following quantities are invariants under a Lorentz transformation: (*a*) $E^2 - c^2B^2$; (*b*) $\mathbf{B} \cdot \mathbf{E}$; (*c*) $H^2 - c^2D^2$; (*d*) $\mathbf{H} \cdot \mathbf{D}$; (*e*) $\mathbf{B} \cdot \mathbf{H} - \mathbf{E} \cdot \mathbf{D}$; (*f*) $c\mathbf{B} \cdot \mathbf{D} + (1/c)\,\mathbf{E} \cdot \mathbf{H}$. [Hint: Show that $\mathbf{B}' \cdot \mathbf{E}' = \mathbf{B} \cdot \mathbf{E}$ etc. by using the field transformations.]

Problem 7.4—The electric field of an infinite line charge of magnitude λ' C/m at rest along the x' axis of the system Σ' is given by $E_x' = B_x' = B_y' = B_z' = 0$;

$$E_y' = \frac{\lambda' y'}{2\pi\varepsilon_0(y'^2 + z'^2)}; \quad E_z' = \frac{\lambda' z'}{2\pi\varepsilon_0(y'^2 + z'^2)}$$

Use this result to determine the magnetic field of a linear current i along the x axis of the Σ-system. What is the electric field in this instance? Compare the magnitudes of the electric and magnetic forces on a test charge q moving with velocity $\mathbf{u}$ at the point x, y, z.

8

THE CLOCK PARADOX

8.1. INTRODUCTION

The clock paradox concerns the readings of an accelerated clock. Ever since the introduction of the theory of special relativity, it has been the subject of great controversy. For the reasons elaborated at the end of this Section, in the author's opinion, in themselves without further assumptions the Lorentz transformations do not say what the readings of an accelerated clock should be. The approach adopted from Section 8.2 onwards will be similar to the approach adopted in Chapter 5, when the variation of mass with velocity was developed. In Chapter 5 the theory of special relativity was used as a heuristic aid to 'discover' the law describing the variation of mass with velocity. This law was then compared with experiment. In this Section, the clock paradox will first be introduced in the conventional way in terms of the Lorentz transformations, so as to give the reader an introduction to the historical background to the controversy, before proceeding to the analysis of the problem in Sections 8.2, 8.3 and 8.4.

Consider two clocks A and B. Let clock B be at rest in a laboratory on the earth and let it be at the origin of an inertial frame Σ. Effects due to the rotation of the earth will be neglected. Let the other clock, clock A, be at rest in a rocket which is moving with uniform velocity v relative to the earth. It will be assumed that at this stage of the journey the rocket is at rest at the origin of an inertial frame Σ'. Let clock B have the same properties relative to the laboratory as clock A has relative to the rocket. Both clocks can be calibrated in terms of some atomic frequency. Let the rocket leave the earth at $t = t' = 0$ when the origins of Σ and Σ' coincide, and let clocks A and B be synchronized at this instant. Let the rocket travel with uniform velocity v until it reaches a clock C in outer space, which is at rest relative to the earth as shown in *Figure 8.1*(a). Let clock C be synchronized with clock B which is at rest on the earth, using light signals as specified by Einstein (cf. Section 3.8). Let the clocks B and C read $(t_L)_{\mathrm{BC}}$ and let clock A on the rocket read $(t'_R)_{\mathrm{BC}}$ when the rocket coincides with clock C.

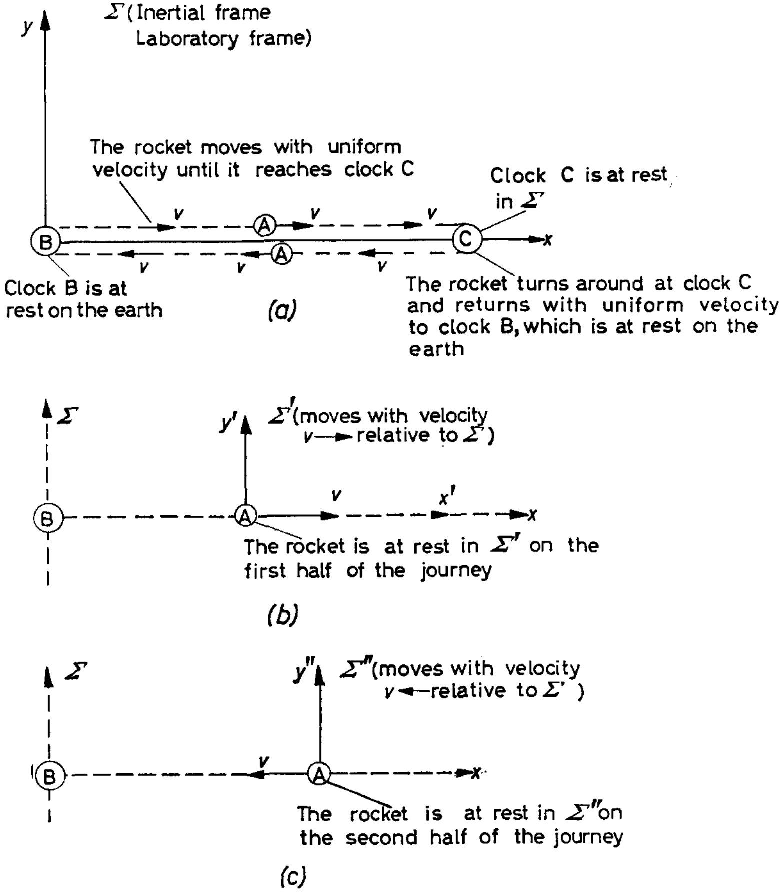

Figure 8.1. Clock A *on the rocket goes on a journey into outer space. The rocket goes with uniform velocity v until it reaches clock* C, *which is at rest relative to the earth; the rocket then turns around and returns to the earth with uniform velocity v*

According to the Lorentz transformations, for the origin of Σ'

$$t = \gamma\left(t' + \frac{vx'}{c^2}\right) = \gamma t'$$

or

$$(t'_R)_{\mathrm{BC}} = (t_L)_{\mathrm{BC}}\sqrt{(1 - v^2/c^2)} \simeq (t_L)_{\mathrm{BC}}(1 - \tfrac{1}{2}v^2/c^2) \qquad (8.1)$$

$$(t_L)_{\mathrm{BC}} - (t'_R)_{\mathrm{BC}} \simeq \tfrac{1}{2}(v^2/c^2)(t_L)_{\mathrm{BC}} \qquad (8.2)$$

Thus according to the Lorentz transformations clock A on the rocket should read less than clock C, which is at rest relative to the earth. The time interval $(t'_R)_{BC}$ measured by the clock on the rocket is a proper time interval, whereas the difference between the readings of clocks B and C, which are both at rest relative to the earth, is not a proper time interval; eqn (8.1) is the expression for time dilation.

We shall now consider what may happen if the rocket turns around at the position of clock C and returns with uniform velocity v back to the earth. In order to turn around the rocket *must undergo an acceleration*. It will be assumed that the rocket turns around quickly and then returns to the earth with uniform velocity v. On the return journey the rocket is at rest in a different inertial frame, say Σ'', from the inertial frame Σ' in which it was at rest on the outward half of the journey. The inertial frame Σ'' is moving with uniform velocity v along the negative Ox axis relative to Σ as shown in *Figure 8.1(c)*. Once the acceleration is over, the transformations of the theory of special relativity can be applied again. The time interval read by clock A on the rocket $(t'_R)_{CB}$ for the time for the rocket to go from C to B, after the acceleration is over, is again a proper time interval, whereas the difference between the readings of clocks C and B when the rocket passes them, namely $(t_L)_{CB}$, is an improper time interval and we should have

$$(t_L)_{CB} = \frac{(t'_R)_{CB}}{\sqrt{(1 - v^2/c^2)}} \tag{8.3}$$

or

$$(t_R)_{CB} = (t_L)_{CB}\sqrt{(1 - v^2/c^2)} \simeq (t_L)_{CB}(1 - \tfrac{1}{2}v^2/c^2) \tag{8.4}$$

What will the readings of clocks B and A be when they coincide at the end of the journey? In his paper of 1905, Einstein[1] wrote:

> From this there ensues the following peculiar consequence. If at the points A and B of K there are stationary clocks which, viewed in the stationary system, are synchronous; and if the clock at A is moved with the velocity v along the line AB to B, then on its arrival at B the two clocks no longer synchronize, but the clock moved from A to B lags behind the other which has remained at B by $\frac{1}{2}tv^2/c^2$ (up to magnitudes of fourth and higher order), t being the time occupied in the journey from A to B.
>
> It is at once apparent that this result still holds good if the clock moves from A to B in any polygonal line, and also when the points A and B coincide.
>
> If we assume that the result proved for a polygonal line is also valid for a continuously curved line, we arrive at this result. If one of two

> synchronous clocks at A is moved in a closed curve with constant velocity until it returns to A, the journey lasting t sec, then by the clock which has remained at rest the travelled clock on its arrival at A will be $\frac{1}{2}tv^2/c^2$ sec slow. Thence we conclude that a balance clock at the equator must go more slowly, by a very small amount, than a precisely similar clock situated at one of the poles under otherwise identical conditions.

Einstein's suggestion is generally referred to as the *clock hypothesis*. According to Einstein, one can add eqns (8.1) and (8.4) to obtain the total time for the journey measured by clock B which is at rest on the earth, namely $(t_L)_{\mathrm{BCB}} = t_L$, and the time $(t'_R)_{\mathrm{BCB}} = t'_R$ measured by clock A which is at rest on the rocket. Following Einstein, one obtains

$$(t'_R)_{\mathrm{BCB}} = (t_L)_{\mathrm{BCB}}\sqrt{(1 - v^2/c^2)} \simeq (t_L)_{\mathrm{BCB}}\left(1 - \frac{1}{2}\frac{v^2}{c^2}\right) \qquad (8.5)$$

or

$$t'_R = t_L\sqrt{(1 - v^2/c^2)} \simeq t_L\left(1 - \frac{1}{2}\frac{v^2}{c^2}\right) \qquad (8.6)$$

Thus, if Einstein's suggestion were correct, the clock on the rocket should read less at the end of the journey than the clock which remained on the earth; this is one example of the clock paradox.

In the above discussion the Lorentz transformations were used directly. The calculation is sometimes carried out using the ideas developed in Section 6.2. For the benefit of those readers who have familiarized themselves with Section 6.2, a brief account of this approach will now be given. The motion of the rocket relative to the earth is represented graphically in *Figure 8.2*. The displacement of the rocket relative to the earth, that is, x, is plotted against $m = ct_L$ using a rectangular co-ordinate system. On the outward half of the journey clock A in the rocket moves with uniform velocity v relative to Σ, and its 'world line' is represented by the line OP which makes an angle $\phi = \tan^{-1} v/c$ with the m axis and an angle $(\pi/2 - \phi)$ with the x axis, as shown in *Figure 8.2*. The rocket coincides with clock C at the 'world point' P. The world line of clock B, which remains at rest on the earth, is represented by the m axis in *Figure 8.2*, and clock B is at the 'world point' Q at the time $t_L/2$ (relative to the earth), when clock A coincides with clock C at the 'world point' P. The distance QP, which is equal to $vt_L/2$, represents the displacement of the rocket at the time $t_L/2$ (relative to the earth). Draw a hyperbola through P parallel to the calibration hyperbola corresponding to $G = -1$. The unit of time for the co-ordinate system Σ' in which the rocket is at rest on the outward half of the journey is proportional to OP whilst the

unit of time appropriate to the laboratory system Σ is proportional to OR. If the length OP represents K units of time on the rocket, then OR represents K units of time in the laboratory. Since $OQ > OR$, OQ represents a longer time in the laboratory system Σ than OP does in the inertial frame Σ' in which the rocket is at rest on the outward half of the journey, so that clock C which is at rest in the laboratory frame should read more than clock A on the

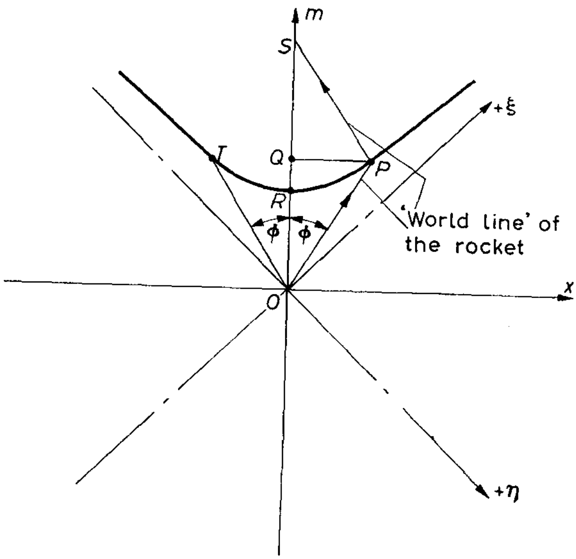

Figure 8.2. The geometrical representation of the journey performed by the rocket, previously illustrated in Figure 8.1(a); the 'world line' of the rocket is represented by the lines OP and PS

rocket, when the rocket is at the world point P. After it has turned around, the 'world line' of the displacement of the rocket relative to the earth, on the return half of the journey when it is at rest in an inertial frame Σ'', is represented by the line PS. The unit of time in the co-ordinate system Σ'' in which the rocket is at rest on the second half of the journey is proportional to OT which is equal to PS. The unit of time in the laboratory system is still proportional to OR. Since $SQ = OQ > OR$, the time interval measured by the clock on the rocket for the return half of the journey, after the acceleration is over, is less than the time interval measured by clocks at rest relative to the earth. According to Einstein (1905) the clock on the rocket would read the time represented by OP (in Σ')

plus PS (in Σ'') which is less than the time represented by OS in Σ, in agreement with eqn (8.6).

So far we have only considered the case in which the rocket always moves with uniform velocity except when it turns around. Einstein's theory can be extended to the general case, when the rocket undergoes a series of accelerations or undergoes a continuous acceleration. In the inertial frame Σ (the laboratory system) for two successive positions of the rocket at (x, y, z, t) and $(x + \delta x, y + \delta y, z + \delta z, t + \delta t)$ we have

$$\left.\begin{aligned} \delta x^2 + \delta y^2 + \delta z^2 - c^2\delta t^2 = \delta s^2 = -c^2\delta\tau^2 \\ \text{or} \qquad \delta\tau = \delta t\sqrt{(1 - u^2/c^2)} \end{aligned}\right\} \tag{8.7}$$

In eqns (8.7), δs is *the* interval between the two events corresponding to the successive positions of the rocket, u is the velocity of the rocket relative to the earth, and $\delta\tau$ is the proper time interval between the two events, so that $\delta\tau$ is equal to the time interval between the two events, measured by a clock at rest in the inertial reference frame in which the rocket is instantaneously at rest. According to the clock hypothesis, the time difference between the two events, measured by the clock inside the rocket, is also equal to $\delta\tau$. Hence, according to the clock hypothesis, the rate of the clock in the rocket is independent of the acceleration of the rocket, and is equal to the rate of a clock of identical construction at rest in the inertial frame in which the rocket is instantaneously at rest. According to the quotation taken from Einstein[1] given earlier, if the rocket is accelerating, eqn (8.7) can be integrated to give, in the general case

$$\frac{1}{ic}\int \mathrm{d}s = \int \mathrm{d}\tau = \int \mathrm{d}t\sqrt{(1 - u^2/c^2)} \tag{8.8}$$

where u is the instantaneous value of the velocity of the rocket relative to the earth. The time of the journey in the laboratory frame is equal to $\int \mathrm{d}t$, whilst the time measured by the clock on the rocket should be equal to $\int \mathrm{d}\tau$. Eqn (8.8) is the mathematical representation of the clock hypothesis. For the special case shown in *Figure 8.1*, when $u = v$ except for the short period when the rocket is turning around, one has

$$\int \mathrm{d}\tau = \sqrt{(1 - v^2/c^2)}\int \mathrm{d}t \quad \text{or} \quad t'_R = \sqrt{(1 - v^2/c^2)}\, t_L \tag{8.9}$$

This is in agreement with eqn (8.6).

Equation (8.8) can be represented geometrically; $\int \mathrm{d}t$ is given by the length OS in *Figure 8.2* using the time scale appropriate to Σ, whilst for a curved 'world line,' representing the motion of the

rocket in the general case, $\int d\tau$ is evaluated along the curved 'world line', using at each stage the time scale appropriate to the inertial reference frame in which the rocket is instantaneously at rest. Both $\int d\tau$ and $\int ds$ depend on the actual 'world line' representing the motion of the moving clock. Both integrals are route dependent, so that neither $d\tau$ nor ds is a perfect differential.

The conclusion that t_R' is generally not equal to t_L is known as the *clock paradox*. Ever since the introduction of the theory of special relativity, the clock paradox has provoked a lot of discussion, and there are still a large number of scientists who disagree with Einstein's conclusion and suggest that if the experiment were actually performed one would find that t_R' would be equal to t_L. The clock paradox is often stated in terms of identical twins, one of whom goes on a journey into outer space and then returns to the earth, whilst the other twin remains at rest on the earth. Will the traveller be younger, that is aged less, than his twin brother who has stayed at home? According to eqn (8.5) he should have aged less. The argument has been revived recently, since there is a strong probability that space travel may become feasible in the not too distant future, though the effect due to the clock paradox, if it exists, will be small since $t_L - t_R' \simeq \frac{1}{2}(v^2/c^2)t_L$, and it will not be possible to accelerate rockets to speeds comparable with the velocity of light, since the fuel requirements would be too heavy. The recent controversy was started, to a large extent, by Dingle[2] and McCrea[3]. Dingle has maintained stoutly against all opposition that if the experiment were performed, there would be no relative aging between the identical twins and t_R' would be equal to t_L.

In order to avoid the emotional impact of expressing the clock paradox in terms of human mortality, the clock paradox will be discussed in terms of the readings of radioactive clocks, and in order to simplify the discussion, the clock paradox will be discussed, initially, in terms of the journey illustrated in *Figure 8.1*, in which the rocket moves with uniform velocity v except when it is turning around. It is accepted by both sides in the dispute that under the *same* conditions a clock will read the same when at rest in both the rocket and the laboratory, e.g. to quote Bondi[4]:

> If one inertial observer makes himself a clock, driven by a clockspring, that ticks twice a second, then any other observer copying exactly this design of clock will have a device for measuring his own time in the same way, with two ticks a second. This is not a theorem but a definition, the definition of seconds for any observer. Instead of a spring driven clock any other device registering time could have been used, for example the spectral line of an atom, the time a specific acid takes to eat through

a specified thickness of steel, the length of a generation of rabbits on a rabbit farm, the decay of a radioactive element, the aging of human beings, and so forth.

The clock paradox will be discussed in terms of the rate of decay of a radioactive element. If one started with N_0 radioactive atoms at rest on the earth and N_0 identical atoms at rest in the rocket at the instant the rocket leaves the earth, then when the rocket returns to the earth, according to the law of radioactive decay, the number of radioactive atoms remaining in the laboratory (denoted N_{LAB}) should be given by

$$N_{\text{LAB}} = N_0 e^{-t_L/T_0} \tag{8.10}$$

where t_L is the time interval measured by a clock at rest in the laboratory. Similarly, the number of radioactive atoms remaining in the rocket when it returns to the earth (denoted by N_{ROCKET}) should be given by

$$N_{\text{ROCKET}} = N_0 e^{-t'_R/T_0} \tag{8.11}$$

where t'_R is the time interval measured by a clock at rest in the rocket. It is assumed that the radioactive substance has the same mean lifetime T_0 when the radioactive substance is at rest in both the laboratory and in the rocket. If, as Einstein suggests, t'_R were less than t_L then one would find experimentally that N_{ROCKET} would be greater than N_{LAB}, whilst if Dingle's view proved correct one would find that N_{ROCKET} would be equal to N_{LAB}. It should be noted that if Einstein's view proved to be correct, then there would be a *physical difference* between the box of radioactive atoms which remained on the earth compared with the box of radioactive atoms which went on the journey in the rocket, since a higher proportion of the radioactive atoms in the box in the laboratory would have decayed than in the box in the rocket. When the rocket turns around it must undergo an acceleration and be transferred from one inertial reference frame to another, so that the clock paradox involves what may or may not happen to a clock which is at rest in a co-ordinate system which is sometimes accelerating relative to the fixed stars.

An attempt is now made to try and state, as objectively as possible, some of the arguments put forward in defence of their viewpoints by the supporters of both sides in the controversy. The main argument of those who say that, if the experiment were actually performed, that t_L would be equal to t'_R is roughly as follows. According to the theory of relativity two co-ordinate systems in uniform relative motion are equivalent, so that clock A in the rocket may be regarded as being at rest with clock B on the earth moving relative to it.

If the acceleration of the rocket has no effect on the clock in the rocket, then the motion of clock B on the earth relative to clock A on the rocket should be precisely the same as the motion of the clock A on the rocket relative to the earth considered previously, except that the velocity of the 'moving' clock is in the opposite direction in the two cases. If it is assumed that clock B on the earth is moving relative to the rocket, then clock B on the earth measures the proper time intervals between the events corresponding to successive positions of the earth relative to the rocket, whereas the clocks 'stationary' in the co-ordinate system in which the rocket is at rest measure improper time intervals. If the arguments used previously are applied, exactly as before, one should now conclude that

$$t_L = \sqrt{(1 - v^2/c^2)}\, t_R' \tag{8.12}$$

In this case t_L should be less than t_R' whereas previously it was concluded that t_L should be greater than t_R'. Both these conclusions cannot be correct. Dingle and his supporters suggest that, *provided* the effects associated with the acceleration of the rocket can be neglected, the way out of the paradox is to conclude that if the experiment were performed one would find that t_L would actually be equal to t_R', so that there would be no relative aging between the two identical twins. This is also a *hypothesis* about the behaviour of an accelerating clock.

Those who uphold the alternative view that, if the experiment were performed, t_R' would be found to be less than t_L, start by pointing out that the clock in the rocket undergoes an acceleration, whereas the clock at rest on the earth does not. (The effects due to the rotation of the earth are being neglected.) When one is in a train or in a car one knows when one undergoes sudden accelerations or de-accelerations. The experiences of somebody undergoing accelerations are different from the experiences of somebody who does not undergo accelerations. The 'experiences' of the clock on the rocket and the clock on the earth are not the same but are different, since one of the clocks, namely the clock on the rocket, undergoes an acceleration. It is suggested that the clock on the accelerating rocket may be changed physically during *or as a result of* the acceleration. It is suggested that, due to the different experiences of the clocks, it would not be unreasonable to find that the readings of the radioactive clocks on the earth and on the rocket, measured in terms of the fraction of undecayed radioactive atoms, differed when the rocket returned to the earth. For the reasons to be elaborated in Sections 8.2, 8.3 and 8.4, they claim that the clock hypothesis, that is eqn (8.8), is correct.

To illustrate the role of the Lorentz transformations, we shall review how the variation of mass with velocity was developed in Chapter 5. In themselves, the Lorentz transformations do not say anything about the mass of a moving body. If a body is moving, then its positions at various times relative to any inertial frame Σ can be determined using rulers and synchronized clocks at rest relative to Σ. Similarly, the successive positions of the body relative to any other inertial frame Σ', moving with uniform velocity relative to Σ, can be determined using rulers and clocks at rest relative to Σ'. It is the co-ordinates and times of the event of the measurement of the position of the moving body at some identifiable point on its path, determined by clocks and rulers at rest in Σ and Σ' respectively, which are related by the Lorentz transformations. By themselves, the Lorentz transformations say nothing about the mass of the body. Velocity is a kinematic quantity which can be determined from the readings of the rulers and clocks at rest relative to Σ and Σ' respectively. Momentum was defined as the product of mass and velocity, the expression for the variation of mass with velocity being unknown at this stage. It was then shown that, if the law of conservation of momentum was to hold in all inertial reference frames, when co-ordinates, times and velocities were transformed using the Lorentz transformations, then mass had to be redefined such that the mass of a particle of rest mass m_0 moving with velocity $\mathbf{u}$ was $m_0/(1 - u^2/c^2)^{\frac{1}{2}}$. The theory of special relativity was used as a heuristic aid to help to select the appropriate law for the variation of mass with velocity. This law was then compared with experiment.

Similarly, if there is a clock in an *accelerating* rocket, the successive positions of the rocket relative to *any* inertial frame Σ can be determined using rulers and synchronized clocks at rest in Σ. Similarly, the successive positions of the rocket relative to any other inertial frame Σ', (which is and continues to move with uniform velocity relative to Σ) can be measured using rulers and synchronized clocks at rest, relative to Σ'. The rocket is accelerating relative to both Σ and Σ'. Consider any identifiable point on the path of the accelerating rocket. It is only the co-ordinates and time of such an event, measured by rulers and clocks which *are, and remain, at rest* relative to Σ and Σ' respectively, which are related by the Lorentz transformations. Without making some further assumption such as the clock hypothesis or the Dingle hypothesis, one cannot deduce what the readings of the clock inside the accelerating rocket should be, since the clock in the rocket is accelerating, from one inertial frame to another. Our approach in

Section 8.2 will be to consider a *gedanken experimente* and use the postulates of the theory of special relativity as a heuristic aid to select the appropriate law describing the behaviour of an accelerated clock. Then in Sections 8.3 and 8.4 these predictions will be compared with what experimental evidence is available. The reader should approach the rest of this Chapter with an open mind, without any preconceived intuitive misconceptions, and base his final opinion on the theoretical and experimental evidence presented to him.

8.2. A *GEDANKEN EXPERIMENTE* ON THE CLOCK PARADOX USING THE DOPPLER EFFECT

The example to be discussed was given by Darwin[5] and expanded by Bondi[4], Scott[6], Builder[7] and others. It is again assumed that a rocket goes on a journey from the earth into outer space with uniform velocity v, that the rocket then turns around quickly and comes back to the earth with uniform velocity v, as illustrated in *Figure 8.1*. Let the time of the journey be measured to be t_L in the inertial reference frame in which the earth is at rest and let it be measured to be t'_R in the co-ordinate system in which the rocket is at rest. The rocket turns around at times $t_L/2$ and $t'_R/2$ relative to the earth and rocket respectively. The K-calculus methods of Appendix 6 are used.

Let signals be sent out from the earth at intervals of $1/\nu_0$, where ν_0 is the frequency of the signals measured relative to the earth. (One can imagine each successive maximum of a monochromatic light wave, emitted by a source of light at rest on the earth as a signal.) Let the frequency of these signals be measured to be $\nu' = \nu_0/K$ relative to the rocket, when they reach the rocket when it is moving away from the earth towards clock C (which is at rest relative to the earth), as shown in *Figure 8.1(b)*. The source of light on the earth is moving away from the observer, who is on the rocket in this instance. Let the rocket re-emit signals without time delay every time it receives a signal from the earth. These secondary signals have a frequency ν' relative to their source (the rocket). If the velocity of light is independent of the velocity of the source of light, the secondary signals from the rocket must be recorded at a frequency $\nu_0 = K\nu'$ by an observer at clock C, at rest relative to the earth, just as if there had been no intermediate stage. In this case the source of light on the rocket is approaching the observer near clock C. This shows that if the frequency of the light received from a source moving away from an observer changes by a factor $1/K$, then, if the velocity of light is the same in all directions, the

frequency of light from a source approaching an observer must change by a factor K. Thus, if a rocket emits signals of frequency ν' (relative to the rocket) and if the frequency of these signals measured by an observer on the earth is ν'/K when the rocket is moving directly away from the earth, then the frequency of the signals received on the earth emitted from the rocket when the rocket is approaching the earth with the same speed should be $K\nu'$. This is an example of the Doppler effect. It will be shown that the value of K agrees with eqn (4.36). (Cf. also Appendix 6*j*.)

Consider the motion of the rocket relative to the earth for the journey shown in *Figure 8.1*. It will be assumed that light signals are sent out from the rocket at intervals $1/\nu'$, where ν' is the frequency of the signals measured relative to the rocket. (One can imagine each successive maximum of a monochromatic light wave, emitted by a source of light at rest on the rocket as a signal.) These signals will travel with a velocity c in empty space and reach the earth at a later time. Let the frequency of light received on the earth, and emitted when the rocket is moving away from the earth be, ν'/K. When the rocket turns around at a time $t_L/2$ relative to the earth, and at a time $t'_R/2$ relative to the rocket, it will be at a distance $vt_L/2$ from the earth, measured in the co-ordinate system in which the earth is at rest. The light emitted by the rocket when it is approaching the earth will have a frequency $K\nu'$ relative to the earth. It will take a time $vt_L/2c$ after the rocket turns around, before the light of the new higher frequency reaches the earth from the rocket. The change in frequency will be observed on the earth at a time $t_L/2 + vt_L/2c$ (measured by a clock at rest on the earth). Since the rocket travels for a time $t'_R/2$ (relative to the rocket) before turning around and the light signals are emitted from the rocket with a frequency ν' (relative to the rocket) during this time interval, the number of signals leaving the rocket between the instant it leaves the earth and the instant it turns around is equal to $\nu' t'_R/2$. The time interval between the arrival of successive signals on the earth (measured in the co-ordinate system in which the earth is at rest) is equal to K/ν'. Thus between the time the rocket leaves the earth and the time $t_L/2 + vt_L/2c$ when the signals of the higher frequency start reaching the earth, the observer on the earth will receive $\nu' t'_R/2$ signals, spaced a time interval K/ν' sec apart, so that

$$\frac{t_L}{2}\left(1 + \frac{v}{c}\right) = \frac{\nu' t'_R}{2} \times \frac{K}{\nu'} \quad \text{or} \quad t_L\left(1 + \frac{v}{c}\right) = Kt'_R \qquad (8.13)$$

The observer at rest on the earth will receive signals at a frequency $K\nu'$ from the time $(t_L/2)(1 + v/c)$ until the time t_L when the rocket

returns to the earth, that is, for a time $(t_L/2)(1 - v/c)$. These signals will be emitted by the rocket between the instant of turning around and the instant when the rocket returns to the earth. During this time interval (which is $t'_R/2$ relative to the rocket) the source on the rocket will emit $\nu' t'_R/2$ signals. Since the time interval between the time of arrival of successive signals on the earth is now equal to $1/K\nu'$ (measured relative to the earth) one has

$$\frac{t_L}{2}\left(1 - \frac{v}{c}\right) = \frac{\nu' t'_R}{2K\nu'} \quad \text{or} \quad t_L\left(1 - \frac{v}{c}\right) = t'_R/K \tag{8.14}$$

Multiplying eqns (8.13) and (8.14), and taking the square root of both sides one obtains

$$t'_R = t_L\sqrt{(1 - v^2/c^2)} \tag{8.15}$$

This is in agreement with eqn (8.6). Dividing eqn (8.13) by eqn (8.14) one finds that K is equal to $[(c + v)/(c - v)]^{\frac{1}{2}}$ in agreement with the relativistic theory of the Doppler effect.

It will now be assumed that light signals are also sent out from the earth to the rocket at a frequency ν_0 (measured in the co-ordinate system in which the earth is at rest). When the rocket is moving directly away from the earth, according to the theory of special relativity the frequency of the light signals received by the rocket is equal to $\nu_0[(c - v)/(c + v)]^{\frac{1}{2}} = \nu_0/K$. This is the frequency of the light signals received by the observer on the rocket until the instant the rocket turns around. Once the rocket has turned around the frequency goes up to $\nu_0[(c + v)/(c - v)]^{\frac{1}{2}} = K\nu_0$. Thus the observer on the rocket receives signals of frequency ν_0/K until the instant the rocket turns around at a time $t_L/2$ (relative to the earth) at a distance $vt_L/2$ away from the earth (the distance being measured in the co-ordinate system in which the earth is at rest). The light reaching the rocket at the instant when it turns around will have left the earth at the time $t_L/2 - vt_L/2c$ (measured relative to the earth). The number of signals emitted from the earth in a time $(t_L/2)(1 - v/c)$ (measured relative to the earth) is equal to

$$(\nu_0 t_L/2)(1 - v/c)$$

This is the number of signals received by the rocket before turning around at a time $t'_R/2$ (relative to the rocket). Since the time interval between the receipt of each signal on the rocket on the outward half of the journey is equal to K/ν_0, one has

$$\frac{t'_R}{2} = \frac{\nu_0 t_L}{2}\left(1 - \frac{v}{c}\right) \times \frac{K}{\nu_0} \quad \text{or} \quad t'_R = Kt_L\left(1 - \frac{v}{c}\right) \tag{8.16}$$

All the light signals emitted from the earth after the time

$$(t_L/2)(1 - v/c)$$

up until the time t_L, that is during a time interval $(t_L/2)(1 + v/c)$, are received on the rocket at a frequency $\nu_0[(c + v)/(c - v)]^{\frac{1}{2}} = K\nu_0$, such that the time interval between each signal is $1/K\nu_0$ (relative to the rocket). Since the number of signals reaching the rocket on the return half of the journey is equal to $\nu_0(t_L/2)(1 + v/c)$,

$$\frac{t'_R}{2} = \nu_0 \frac{t_L}{2}\left(1 + \frac{v}{c}\right) \times \frac{1}{K\nu_0} \quad \text{or} \quad t'_R = \frac{t_L}{K}\left(1 + \frac{v}{c}\right) \qquad 8.17)$$

Multiplying eqns (8.16) and (8.17) and taking the square root of both sides, one obtains

$$t'_R = t_L\sqrt{(1 - v^2/c^2)} \qquad (8.18)$$

This is precisely the same as before [cf. eqn (8.15)]. Thus according to the *Gedanken Experimente* both the observer on the earth and the observer on the rocket would agree that t'_R was less than t_L. One can see precisely how the experiences of the observers on the earth and the rocket differ as far as the receipt of light signals are concerned. The observer on the rocket observes a change in the frequency of the light coming from the earth at the instant the rocket turns around, that is, half way through the journey. On the other hand, the observer on the earth does not observe a change in the frequency of the light coming from the rocket until after half time, but has to wait until light emitted from the rocket, after it turns around, has time to reach him before he measures a change of frequency. It is generally accepted that once the photons constituting light leave the source of light, they travel with the velocity of light and their frequencies and energies relative to any inertial reference frame are unaffected by any subsequent changes in the motion of the source of light. It is also generally accepted that, if an observer changes his state of motion, he immediately observes a change in the energy and frequency of a photon (cf. Section 5.8.3). *If* one makes these assumptions, then it is difficult to find flaws in the above interpretation of the *Gedanken Experimente.* The conclusion that if the experiment were performed, one would find

$$t'_R = t_L\sqrt{(1 - v^2/c^2)}$$

is certainly consistent with relativistic principles.

In order to obtain the answer $t'_R = t_L$ in our *Gedanken Experimente* one could use the classical ether formulae for the Doppler

effect, but it was shown in Section 3.10 that these formulae were not consistent with the experiment of Ives and Stilwell (1938). If the relativistic formulae for the Doppler effect are accepted, one would have to make some other assumptions in order to obtain the answer $t'_R = t_L$. For example, Dingle[8] has suggested that the frequency of the light reaching the earth from the rocket might change at the instant the rocket turns around. However, such a change in the frequency of the light would imply a change in the energy of the photons, and this would imply instantaneous action at a distance, which is not consistent with the theory of special relativity, since, according to relativistic mechanics, energy changes cannot be transmitted with a velocity exceeding the velocity of light.

The above interpretation of the *gedanken experimente* strongly favours the clock hypothesis, eqn (8.8). We shall now proceed to consider possible experimental checks of the clock hypothesis.

8.3. A POSSIBLE EXPERIMENTAL CHECK OF THE CLOCK PARADOX

It was pointed out in Section 8.1 that if one started with N_0 radioactive atoms at rest on the earth and N_0 radioactive atoms at rest in the rocket at the instant the rocket leaves the earth on the journey illustrated in *Figure 8.1*, then according to the law of radioactive decay

$$N_{\text{LAB}} = N_0 e^{-t_L/T_0} \tag{8.19}$$

$$N_{\text{ROCKET}} = N_0 e^{-t'_R/T_0} \tag{8.20}$$

where N_{LAB} and N_{ROCKET} are the number of undecayed radioactive atoms left on the earth and on the rocket respectively at the instant the rocket returns to the earth; t_L and t'_R are the times for the complete journey measured by clocks on the earth and the rocket respectively, and T_0 is the mean life of the radioactive atoms measured in the co-ordinate system in which the radioactive atoms are at rest. If t'_R were less than t_L, then one should find experimentally that N_{ROCKET} would be greater than N_{LAB}. A possible radioactive clock would be a box of π-mesons which should decay with a mean life of $2{\cdot}55 \times 10^{-8}$ sec, when they are at rest in both the rocket and in the laboratory. Instead of putting the π-mesons in a rocket, it is suggested that the π-mesons should be sent around a closed path (*in vacuo*) in the laboratory. Fairly monoenergetic beams of π-mesons of kinetic energy $\sim$100 MeV or more are now

available. The possibility of using accelerator produced π-mesons to check the clock paradox was suggested by Martinelli and Panofsky[9], Cochran[10], and others.

The experiment suggested would not be exactly the same as the rocket experiment, since N_{LAB} and N_{ROCKET} would not be measured at the same time. The proposed experiment will, however, be interpreted by comparing it with the rocket experiment, and it is hoped that it will be accepted as being equivalent to the rocket experiment.

The experimental arrangement suggested is shown in *Figure 8.3*. It is suggested that a collimated beam of π-mesons be passed through the scintillation counter C_1, and after deflection in a

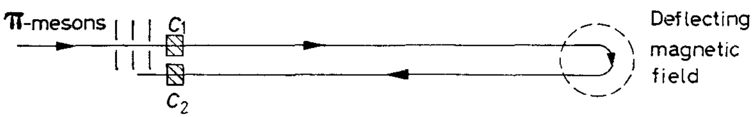

Figure 8.3. A possible experimental arrangement to test the clock paradox.

magnetic field passed through a second scintillation counter C_2 placed close to C_1 and coincidences between C_1 and C_2 detected. If necessary the π-mesons could be sent back through the same crystal C_1. If the intensity of the π-meson beam were low enough, the π-mesons, would generally come one at a time and the number of chance coincidences would be small. The μ-mesons arising from $\pi-\mu$ decays would not generally be emitted in the same direction as the parent π-mesons so that π-mesons, which would undergo $\pi-\mu$ decay before reaching C_2, would not give rise to coincidences between counters C_1 and C_2, except when the μ-meson happened to be emitted within the solid angle subtended by C_2. (A correction could be applied for these stray coincidences.) The number of π-mesons which started the journey would be given by the number of single counts of counter C_1 (denoted by N_0), whilst the number of π-mesons which completed the journey (denoted by N_{ROCKET}) would be given by the number of coincidences between C_1 and C_2 (after correcting for any focusing losses, stray coincidences, etc.). Instead of starting with N_0 π-mesons moving along the path shown in *Figure 8.3* and observing how many remained when they reached C_2, which would be directly equivalent to sending them in a rocket, it is suggested that N_0 successive π-mesons be observed going one at a time around the path shown in *Figure 8.3*, and the number reaching C_2 observed. If 'monoenergetic' π-mesons

were used, then N_0 successive experiments using one π-meson at a time should be equivalent, when added together, to starting with N_0 π-mesons and doing the experiment once only. This experiment is equivalent to sending N_0 π-mesons on a journey in a rocket and counting how many did not decay before the end of the journey. This experiment gives N_{ROCKET}.

The time delay in the laboratory between the light pulses in C_1 and C_2 due to π-mesons, which complete the path shown in *Figure 8.3* and pass through both scintillation counters, should be measured. Such a time interval could be measured accurately by a time to pulse height converter of the Green and Bell[11] (1958) type, which could measure time intervals down to 10^{-10} sec. For a total path length of four metres traversed at a velocity of 2×10^8 m/sec the time interval would be 2×10^{-8} sec. This time delay gives t_{LAB}, the time of flight of the π-mesons (or 'rocket') from C_1 to C_2 measured relative to the laboratory. Since the mean life of stationary charged π-mesons has been shown to be $2{\cdot}55 \times 10^{-8}$ sec, one can then use eqn (8.19) to calculate what number N_{LAB} of an initial number N_0 of π-mesons stationary in the laboratory would remain after a time interval t_{LAB}, the time of the 'rocket' journey from C_1 to C_2 measured relative to the laboratory.

The experiment consists of comparing the values of N_{ROCKET} and N_{LAB}. If Dingle's view were correct one would expect N_{LAB} to be equal to N_{ROCKET}, corresponding to equal time intervals on the clocks at rest in the rocket and in the laboratory, that is, $t_L = t'_R$ as determined from eqns (8.19) and (8.20). On the other hand, if Einstein's view were correct, N_{ROCKET} should be greater than N_{LAB} corresponding to $t_L > t'_R$. If t_L were equal to $\gamma t'_R$ [as predicted by eqn (8.6)], where $\gamma = (1 - v^2/c^2)^{-\frac{1}{2}}$, v being the velocity of the π-mesons, then t_L/t'_R should be equal to $1{\cdot}73$ for π-mesons of kinetic energy 100 MeV and $4{\cdot}5$ for π-mesons of kinetic energy 500 MeV. According to Dingle's view the ratio should be equal to unity in both cases. The experiment suggested should be accurate enough to decide between the two theories.

It was described in Section 3.10.4, how experiments such as those of Durbin, Loar and Havens[12] confirmed the dilation of lifetimes, when π-mesons travel in straight lines. According to the clock hypothesis, one should measure the same dilation of lifetimes, relative to the laboratory, when π-mesons travel in curved paths, for example in a circle in a magnetic field or in the path shown in *Figure 8.3*, even if the π-mesons do return to their starting point. An observer travelling with the π-mesons should still, however, measure their lifetime at rest to be $T_0 = 2{\cdot}55 \times 10^{-8}$ sec.

8.4. EXPERIMENTS ON THE TEMPERATURE DEPENDENCE OF THE MÖSSBAUER EFFECT

It was suggested by Sherwin[13] that experiments on the temperature dependence of the Mössbauer effect in ^{57}Fe are a check of the clock paradox. The interpretation of the experiment given by Sherwin is somewhat similar to the *Gedanken Experimente* described in Section 8.2. In this *Gedanken Experimente* it was assumed that a rocket going on a journey into outer space and back sends light signals back to the earth, where the number of signals and the time intervals between successive signals are recorded. A short account of the Mössbauer effect is given in Appendix 5. Under certain conditions, when an excited nucleus forms part of a crystal lattice when it decays, the γ-ray may be emitted without nuclear recoil. Similarly, under certain conditions, when it forms part of a crystal lattice, a nucleus may absorb a γ-ray of the appropriate energy without recoil. Recoilless emission followed by recoilless absorption leading to nuclear resonance fluorescence is known as the Mössbauer effect. It is illustrated in Appendix 5 how a measurement of the Mössbauer effect affords, under certain circumstances, a very accurate method of measuring the difference between two frequencies, for example using ^{57}Fe it is possible to measure a change of frequency less than 1 part in 10^{12}. If the temperature of the source or absorber in the experimental arrangement, discussed in Appendix 5 and illustrated in *Figure A5.4*, is changed, then there is a change in the counting rate of the detector, due to the changes in the amount of recoilless emission (or absorption) due to the change in the temperature and hence the velocity of the source (or absorber) atoms.

The atoms in a crystal are not at rest but are vibrating relative to the laboratory, and are therefore moving relative to the laboratory when the γ-rays are emitted. The mean square velocity of vibration depends on the temperature of the crystal. According to the theory of the Doppler effect the vibrations of the excited nucleus lead to changes in the energy (and frequency) of the photons emitted. Now, light is emitted in the form of discrete photons or quanta and not as a continuous wave motion. Sherwin suggests that this is a point of detail. For example, he writes:

> The waves representing the photon obey Maxwell's equations, and are propagated in exact accordance with requirements of special relativity even though they are detected by the usual discrete quantum events. Furthermore, in quantum mechanics, the excitation and de-excitation of

states by electromagnetic waves is calculated as if the process were continuous, that is, in a manner essentially identical to that used for macroscopic resonant structures. It is only in the matter of interpretation and observation of the calculated state amplitudes that the characteristic discrete quantum effects enter.

The 'waves' representing a single photon can be interpreted in the same way as the de Broglie waves for a single particle. The wave theory of light can be used to estimate the probability that a measurement on *one* photon will give a particular experimental result. Following Sherwin, the emission of a γ-ray will be interpreted using the classical wave theory, keeping the above quantum mechanical interpretation in mind. According to classical theory, during the lifetime of the excited state the waves are emitted continuously. When discussing the 14·4 keV level in ^{57}Fe, Sherwin writes:

> Since the centre frequency ν_0 is $3{\cdot}46 \times 10^{18}$ c/sec, and since the mean lifetime of the excited state is $1{\cdot}4 \times 10^{-7}$ sec, there are about 5×10^{11} complete cycles in the (intense part of the) wave train representing the photon.

For a crystal the frequency of vibration of ions is $\sim 10^{13}$ c/sec, so that a nucleus vibrates about its mean position many times during the lifetime of the excited state in ^{57}Fe. According to classical wave theory the motion of the decaying nucleus at any instant affects the frequency of the waves emitted at that particular instant. For simplicity it will be assumed that the excited nucleus vibrates inside the crystal in one dimension only, namely, along the x axis. It will be assumed further that the ^{57}Fe nucleus moves with uniform velocity v relative to the laboratory, turns round and moves back with uniform velocity v until it turns around again. Consider one complete oscillation as shown in *Figure 8.4*. Classically, one can imagine the source nucleus emitting waves continuously, precisely as in the *Gedanken Experimente* discussed in Section 8.2. The moving source corresponds to the rocket. The signals are emitted at a frequency ν_0 relative to the source (rocket). The receiving nucleus measures the frequency of the waves representing the photons emitted by the source nucleus, using its own characteristic frequency as a standard of comparison. It is being assumed for the present that the receiving (absorbing) nucleus is at rest. If the receiving (absorbing) nucleus were at position 2 in *Figure 8.4*, then it would correspond precisely to the *Gedanken Experimente* discussed in Section 8.2 and illustrated in *Figure 8.1*. However, source and absorber do not coincide at the beginning and at the end of the journey, but the absorber is displaced a distance x_0

from position 2 as shown in *Figure 8.4*. According to the principle of the constancy of the velocity of light, the velocity of the waves from the source is independent of the velocity of the source, so that the time light takes to travel the *extra* distance x_0 is always x_0/c. Hence, the time interval between the receipt of the first and last signal from the 'rocket' on the path shown in *Figure 8.4* is the same,

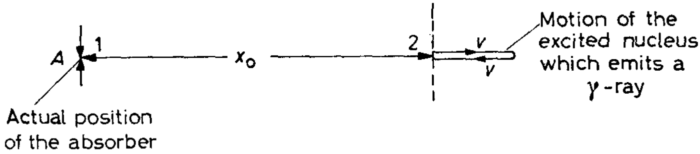

Figure 8.4. A simplified diagram of an experiment on the temperature dependence of the Mössbauer effect. It is assumed that the emitting nucleus is moving back and forth with uniform velocity v; its path is somewhat similar to the path of the rocket shown in Figure 8.1(a)

whether the absorber is at position 2 or position 1. According to the theory of the *Gedanken Experimente* developed in Section 8.2:

$$t'_R = t_L\sqrt{(1 - v^2/c^2)} \tag{8.21}$$

where t'_R and t_L are the times for the journey, illustrated in *Figure 8.4* measured by clocks at rest on the 'rocket' (i.e. source) and on the earth respectively. Eqn (8.21) is also consistent with the use of eqn (8.8). If, during the lifetime of the excited state, the excited nucleus goes back and forth many times on the journey shown in *Figure 8.4*, then, since the journey is merely a series of repetitions of the one shown in *Figure 8.4*, one has

$$t'_R = t_L\sqrt{(1 - v^2/c^2)} \tag{8.22}$$

where now t'_R and t_L are the lifetime of the excited state measured relative to the 'rocket' (source) and laboratory (absorber) respectively. Let N_0 be the number of maxima of the wave train emitted by the source nucleus. The average frequency of the waves relative to the laboratory is given by

$$\nu_L = \frac{N_0}{t_L}$$

whereas the average frequency of the waves relative to the 'rocket' (source nucleus) is given by

$$\nu'_R = \nu_0 = \frac{N_0}{t'_R}$$

Hence

$$\nu_L = \nu'_R\sqrt{(1 - v^2/c^2)} = \nu_0\sqrt{(1 - v^2/c^2)}$$

where ν_0 is the mean frequency of the emitted γ-ray in the inertial frame in which the excited nucleus is at rest. If the excited nucleus is vibrating in 3 dimensions with non-uniform velocity, then one should use eqn (8.8), namely

$$\int d\tau = \int dt\sqrt{(1 - u^2/c^2)}$$

or

$$\int dt'_R = \int dt_L\sqrt{(1 - u^2/c^2)}$$

where u is the instantaneous value of the velocity of the source nucleus relative to the laboratory.

Let $\Delta\nu_s$ be the *shift* in the frequency of the emitted γ-ray when the nucleus is in a lattice at a temperature $T_S^\circ K$. We have,

$$\frac{\Delta\nu_s}{\nu_0} = (1 - v^2/c^2)^{\frac{1}{2}} - 1 \simeq -v^2/2c^2$$

For the general case of non-uniform velocity, v^2 would be the mean square velocity of the excited nucleus. If M is the mass of the nucleus, according to classical kinetic theory, we would have

$$\tfrac{1}{2}Mv^2 = 3(kT_S)/2 \quad \text{or} \quad v^2 = \frac{3kT_S}{M}$$

Hence,

$$\frac{\Delta\nu_s}{\nu_0} = -\frac{3kT_S}{2Mc^2} = -2{\cdot}4 \times 10^{-15}T_S$$

where T_S denotes the temperature of the source.

So far it has been assumed that the nuclei in the absorber are at rest. In practice, they are also vibrating, their mean square velocity depending on T_A the temperature of the absorber. The shift in the resonance frequency of absorption is given by

$$\frac{\Delta\nu_A}{\nu_0} = -2{\cdot}4 \times 10^{-15}T_A$$

Hence the frequency difference $\Delta\nu$ between the source resonant

frequency and the absorber resonant frequency is given by

$$\frac{\Delta\nu}{\nu_0} = -2{\cdot}4 \times 10^{-15}\Delta T \tag{8.23}$$

where $\Delta T = T_S - T_A$. To quote Sherwin again:

> If the source is at a higher temperature than the absorber, then $\Delta\nu$ is negative, that is, the source has a lower frequency than the absorber. If resonance absorption is to occur, the absorber must be given a small velocity away from the source. By contrast, if the source is at a lower temperature than the absorber, then for resonance to occur, it is found necessary that the absorber move toward the source.

According to Pound and Rebka[14] for iron near room temperature, the numerical coefficient in eqn (8.23) should be about 0·9 times the classical value, corresponding to a Debye temperature of 467°. Hence eqn (8.23) should read

$$\frac{\Delta\nu}{\nu_0} = -2{\cdot}21 \times 10^{-15}\Delta T$$

Pound and Rebka found experimentally that,

$$\frac{\Delta\nu}{\nu_0} = -(2{\cdot}09 \pm 0{\cdot}24) \times 10^{-15}\Delta T$$

if both the source and absorber were near room temperature. The agreement between the calculated and experimental results is satisfactory. Thus the temperature dependence of the Mössbauer effect can be accounted for in terms of eqn (8.8), or the *Gedanken Experimente* discussed in Section 8.2.

Hay, Schiffer, Cranshaw and Egelstaff[15] placed a ^{57}Co source near the centre of a rotating wheel. A thin ^{57}Fe iron absorber was placed around the circumference of the wheel. The transmission of the absorber was measured for various angular velocities, using a counter at rest just outside the wheel, to detect the 14·4 keV γ-rays. The transmission of the absorber was found to increase as the angular velocity increased, indicating a shift in the characteristic frequency of the absorber. Since the line shape for the absorber at rest was known experimentally, the magnitude of the frequency shift could be estimated, and it was found to agree with the frequency shift calculated using eqn (8.8) in the laboratory system. If the linear velocity of the absorber is v, then the characteristic frequency relative to the laboratory is given by

$$\nu_A = \nu_0\sqrt{(1 - v^2/c^2)}$$

where in this case the moving absorber is equivalent to the moving rocket. The experiment can also be interpreted in terms of the theory of general relativity in the rotating system (cf. Burcham[16], and see Section 9.8).

These results on the Mössbauer effect are extremely interesting in that they can be interpreted in terms of eqn (8.8) as described above. However, the interpretation of the experiments in this way is based on the quantum mechanical statistical interpretation of photon 'waves'. One has to assume this interpretation to be correct in order to prove that the experimental results are a check of the clock paradox. For this reason it is still worth while performing the π-meson experiments described in Section 8.3, since these experiments approximate more closely to the rocket experiment illustrated in *Figure 8.1*. These experiments would also be easier to perform than putting clocks in satellites or rockets as has been suggested.

8.5. DISCUSSION OF THE CLOCK PARADOX

The clock paradox has been viewed as an extension of the theory of special relativity, since it involves what may happen when measuring apparatus such as clocks are accelerated relative to the fixed stars. The discussion was confined largely to the simple example illustrated in *Figure 8.1*, in which a rocket travels with uniform velocity v relative to the earth, turns around and comes back with the same uniform velocity v relative to the earth. It was shown in Sections 8.2 and 8.3 that the view that, if the experiment were performed, one would find that t'_R would be less than t_L is not inconsistent with relativistic principles and with the experiments that have already been performed with π-mesons and μ-mesons. It has been shown by Tolman[17] and Møller[18] that the result $t'_R = (1 - v^2/c^2)^{\frac{1}{2}} t_L$ is consistent also with the theory of general relativity (cf. Section 9.5).

Both the equation

$$\int d\tau = \int \sqrt{(1 - u^2/c^2)}\, dt \tag{8.8}$$

and the theory of general relativity have been applied to calculate t'_R and t_L when the motion of the rocket is non-uniform. For a full discussion the reader is referred to Bondi[4], Cochran[19] and Scott[6]. It does not always turn out that t'_R is always less than t_L. For example, Cochran writes:

> It is amusing to find, in view of the controversy on the aging of space travellers, that in the simplest form of space travel, the traveller ages *most*! For a clock Z which is thrown up in a uniform gravitational

field, moves in a straight line, and returns *under its own weight* will record a greater time than a resting clock Y with which it is synchronized at the point of release· · ·

Let us imagine a rocket launching platform above the earth's atmosphere, yet firmly attached to the earth, and two space travellers sent off simultaneously, one in orbit around the earth and the other vertically upwards. The latter is given a velocity less than the escape velocity, just sufficient to allow him to return at the instant when the former has completed his first orbit. It would be found that the traveller in the satellite had aged less than a member of the launching crew, while the other traveller has aged more!

The above example will be discussed in Section 9.6, after discussing the effects of gravitational fields on the rates of clocks, that is the gravitational red shift.

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PROBLEMS

Note: When answering the numerical examples which follow, the reader should assume that the clock hypothesis

$$\tau = \int_{t_1}^{t_2} \sqrt{(1 - u^2/c^2)}\, dt$$

is correct; τ is the time interval measured by a clock moving with instantaneous velocity $\mathbf{u}$ relative to an inertial frame in which the same time interval is measured to be $t_2 - t_1$.

Problem 8.1—Einstein wrote: '$\cdots$ the theory becomes a valuable heuristic aid in the search for general laws of nature'.

Discuss what is meant by this statement, which Einstein made about the theory of relativity. Give examples of the use of this method.

Problem 8.2—Discuss the role the velocity of light plays in the theory of special relativity as the limiting velocity for the propagation of signals. (References: Grünbaum[20] and Schwartz[21]).

Problem 8.3—Review the arguments, both for and against the view that, if one of two identical twins went on a journey into outer space and returned to the earth, he would have aged differently from his twin brother who stayed at home.

Problem 8.4—A rocket accelerates quickly and then moves with uniform velocity relative to the solar system, until it reaches a star 8 light years away. A clock on the rocket records the time for the journey as 6 years. Calculate (*a*) the speed of the rocket relative to the earth and (*b*) the time of the journey relative to the earth. (*c*) If the rocket turns around quickly and returns to the earth with the same velocity as on the outward journey compare the total time for the journey relative to the earth and relative to the rocket.

Problem 8.5—On her 29th birthday a lady physicist concludes that she would like to remain 29 for at least 10 years. She decides to go on a journey into outer space with uniform velocity. What is the minimum speed she must move relative to the laboratory so that she can return 10 years later (relative to the laboratory) and still say, quite truthfully, that she is only 29?

Problem 8.6—An astronaut goes on a 24 hour journey in a satellite, which is in a circular orbit around the earth. If the velocity of the satellite relative to the earth is 10^4 m/sec, find how much less the astronaut will have aged than a person who remains on the ground. You may neglect effects associated with the change in gravitational potential at the altitude of the satellite.

Problem 8.7—A total of 2,000 charged π-mesons are created at the origin of an inertial frame. Half of the π-mesons remain at rest at the origin, whilst the other half go on a journey with a uniform speed of $0{\cdot}995c$. After a path length of 15 m, the travelling π-mesons are deflected in a magnetic field so that they return back to the origin. (*a*) How many of the moving π-meson should survive the journey? (*b*) How many of the stationary π-mesons are left at the origin when the travelling π-mesons return? Take the mean lifetime of charged π-mesons to be equal to $2{\cdot}5 \times 10^{-8}$ sec, when they are at rest.

Compare the above experiment with the travelling twin experiment and show that the two are equivalent.

Problem 8.8—The most comfortable way of going on a space flight would be in a rocket which moved with proper acceleration g, that is, with acceleration g relative to the inertial reference frame in which the rocket is instantaneously at rest. The astronaut should then feel perfectly at home.

This is an example of hyperbolic motion discussed in Section 5.4.1, and in *Problem 4.10.* Relative to the earth, if the rocket starts from rest at $t = \tau = 0$, one has, as before, u the velocity of the rocket, $\gamma = (1 - u^2/c^2)^{-\frac{1}{2}}$ and x the distance travelled by the rocket relative to the earth given by

$$u = gt\left\{1 + \frac{g^2t^2}{c^2}\right\}^{-\frac{1}{2}}; \quad \gamma = \left\{1 + \frac{g^2t^2}{c^2}\right\}^{+\frac{1}{2}}$$

and

$$x = \frac{c^2}{g}\left[\left\{1 + \frac{g^2t^2}{c^2}\right\}^{\frac{1}{2}} - 1\right]$$

where t is the time of the journey relative to the earth. Assume that τ, the time on the rocket is given by

$$\tau = \int \sqrt{(1 - u^2/c^2)}\, \mathrm{d}t = \int \mathrm{d}t/\gamma$$

and show that

$$t = (c/g) \sinh\left\{\frac{g\tau}{c}\right\} \quad \text{or} \quad \tau = \frac{c}{g} \ln\left\{\frac{gt}{c} + \left(\frac{g^2t^2}{c^2} + 1\right)^{\frac{1}{2}}\right\}$$

Hence, show that

$$u = c \tanh (g\tau/c); \ \gamma = \cosh (g\tau/c)$$

$$x = \frac{c^2}{g}\left[\cosh\left(\frac{g\tau}{c}\right) - 1\right]$$

These equations enable one to relate the velocity of the rocket, and the distance the rocket has travelled relative to the earth in terms of either t or τ, and they also enable one to relate t and τ.

If the rocket goes with proper acceleration g for (*a*) one year, (*b*) 10 years, (*c*) 100 years relative to the earth, how far will it go and what is the time relative to the rocket? Take g to be 32 ft/sec^2.

Problem 8.9—A space-ship leaves the earth and moves with constant proper acceleration $g = 32$ ft/sec^2 relative to the inertial frame in which it is instantaneously at rest, until the space-ship is half way to a star. It then decelerates with proper deceleration g until it reaches the star. It turns around and returns with proper acceleration g for the first half of the return journey and decelerates with proper deceleration g for the last half of the return journey. If the total time for the journey is (*a*) 40 years and (*b*) 4,000 years (relative to the earth) how far away is the star, and what is the time for the complete journey relative to the rocket? (Comment: This problem and the previous one are taken from Bondi[4] where a full discussion of the problem is given.)

Problem 8.10—Give a short account of the Mössbauer effect. Describe how the effect depends on the temperatures of the source and the absorber. Explain how observations of the temperature dependence of the Mössbauer

effect have been interpreted in terms of the clock hypothesis that $\tau = \int \sqrt{(1 - u^2/c^2)}\, dt$.

Problem 8.11—Use geometrical methods to show that t'_R, the time for the journey of the rocket represented by the 'world line' *OPS* in *Figure 8.2* (page 237) and measured by a clock on the rocket, is equal to $t_L\sqrt{1 - v^2/c^2}$, where v is the velocity of the rocket and t_L is the time for the journey measured by a clock on the earth. (Hint: $t'_{OP} = OQ \sec \phi/g'_2$ and $t''_{PS} = QS \sec \phi/g''_2$, where $\tan \phi = v/c$ and $g'_2 = g''_2 = \gamma\sqrt{1 + (v^2/c^2)}$.)

9

THEORY OF GENERAL RELATIVITY

9.1. INTRODUCTION

The theory of special relativity is confined to the discussion of physical phenomena relative to inertial reference frames. It is not applicable in non-inertial frames of reference. It was shown in Section 1.5 that it is necessary to introduce 'fictitious' inertial forces if one wishes to apply Newton's laws of motion in a co-ordinate system accelerating or rotating relative to an inertial frame. It is possible to measure one's rotation relative to the fixed stars from experiments carried out in a closed laboratory. Newton believed that absolute rotations could be measured. After developing the special theory of relativity, Einstein strove for ten years to develop a general theory of relativity.

Einstein felt that it should be possible to express the laws of physics in a covariant form for all reference frames, that is, the mathematical form of the laws of physics should be the same in all frames of reference whether they are accelerating or not. This is the *principle of covariance*. The general mathematical equations might reduce to simpler mathematical forms in inertial reference frames, and they should be consistent with the theory of special relativity in inertial reference frames.

In order to develop the full theory, Einstein found it necessary to use tensor analysis. It would be beyond the scope of the present book to discuss the general theory comprehensively. The discussion is therefore confined to an account of the *principle of equivalence*, and to a few applications of it. This gives the reader a little insight into the general theory, and illustrates some of the limitations of the special (or restricted) theory of relativity. The reader should not expect a complete elucidation of Einstein's theory of general relativity from the following discussion, which is no more than a popular account, designed to help the reader to tackle the more comprehensive books by Eddington, Tolman, Møller, Fock, etc.

9.2. THE PRINCIPLE OF EQUIVALENCE

Let an astronaut be in a space-ship which is at rest in an inertial frame Σ in a region of outer space where there is no gravitational

field, as shown in *Figure 9.1(a)*. There are no windows in the space-ship so that the astronaut cannot look out, and it is soundproof so that he cannot hear anything. Everything inside the space-ship is in a state of weightlessness. Now let a signal from the earth fire some small rockets so that the space-ship starts moving with uniform

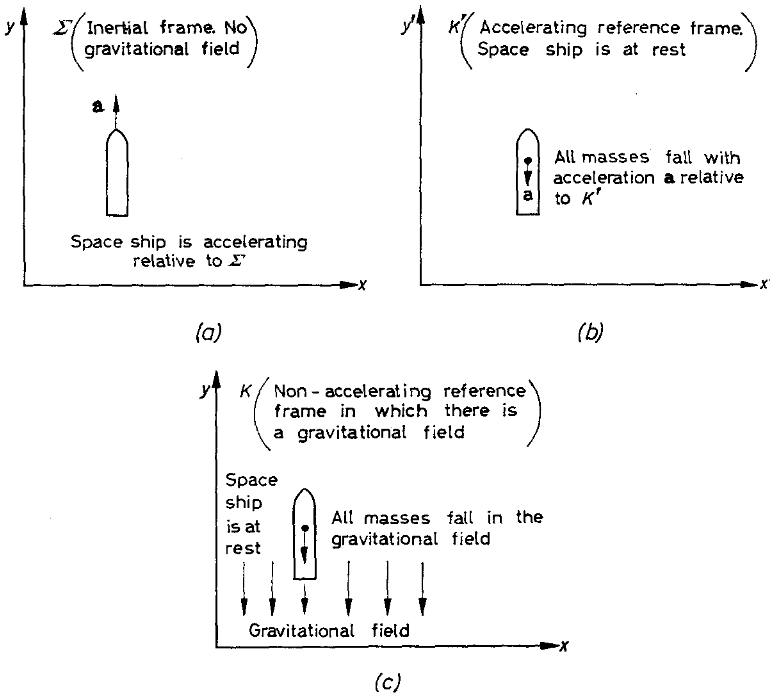

Figure 9.1. The space-ship has uniform acceleration **a** *relative to the inertial reference frame* Σ*. The accelerating reference frame K′ is the reference frame in which the space-ship is at rest; K′ has an acceleration* **a** *relative to* Σ*. All bodies not acted upon by any forces accelerate downwards in K′. According to the principle of equivalence, measurements carried out in K′ yield the same results as experiments carried out in the non-accelerating reference frame K shown in (c), in which there is a gravitational field, opposite in direction to* **a** *and of such a strength that the acceleration due to the force of gravity is numerically equal to* **a**

acceleration **a** relative to Σ. Consider what happens from the viewpoint of the reference frame K', shown in *Figure 9.1(b)*, in which the space-ship is always at rest. The astronaut is no longer in a state of weightlessness. If he holds up an apple and lets it go, the apple will move with uniform velocity relative to the inertial reference frame Σ, but relative to the space-ship it falls downwards

with uniform acceleration. Since all free bodies not acted upon by any forces move with uniform velocity relative to Σ, they all appear to fall with the *same* acceleration relative to the space-ship. If a spring balance, with a weight attached to it, is attached to the roof of the space-ship, the weight would remain at rest (or move with uniform velocity) relative to Σ, unless the spring were under sufficient tension to give the weight the same acceleration as the space-ship has relative to Σ. If the acceleration of the space-ship were equal to g, the acceleration due to gravity on the surface of the earth, the conditions in the space-ship would be similar to the conditions in a space-ship at rest on the surface of the earth. How should the astronaut interpret the results in the accelerating space-ship, if he did not know that rockets were accelerating the space-ship? He would be quite as justified as Newton was in concluding from the observations on a falling apple, that he was in a gravitational field. The astronaut could interpret the tension in the spring balance as an effect due to the weight's being attracted in a gravitational field. In fact, the astronaut could not tell the difference between times when his space-ship was accelerated relative to the fixed stars in a gravitational free region, and times when the space-ship was at rest (or moving with uniform velocity) relative to the fixed stars and when massive bodies were passing underneath the space-ship giving rise to a gravitational field.

In 1911, Einstein proposed that a co-ordinate system K', illustrated in *Figure 9.1(b)* accelerating with uniform acceleration relative to the fixed stars, and a system K, illustrated in *Figure 9.1(c)*, at rest in a homogeneous gravitational field, are exactly equivalent. Experiments carried out under the same conditions in K and K' should give the same numerical results. This is the ***principle of equivalence***. To quote Einstein[1]:

> But we arrive at a very satisfactory interpretation of this law of experience if we assume that the systems K and K' are physically exactly equivalent, that is, if we assume that we may just as well regard the system K as being in a space free from gravitational fields, if we then regard K as uniformly accelerated. This assumption of exact equivalence makes it impossible for us to speak of the absolute acceleration of the system of reference, just as the usual theory of relativity forbids us to talk of the absolute velocity of a system; and it makes the equal falling of all bodies in a gravitational field a matter of course.

Since all bodies not acted upon by any forces should continue to move with uniform velocity relative to an inertial reference frame whatever their inertial masses, they should all have the same acceleration relative to an accelerated reference frame. Hence, according

to the principle of equivalence all bodies should fall with the same acceleration in a homogeneous gravitational field. It was shown in Section 1.3 that, if the acceleration of bodies towards the centre of the earth is the same, then the inertial mass and the gravitational mass of a body are proportional to each other. [If M is the gravitational mass of the earth and m_g and m_I are, respectively, the gravitational mass and the inertial mass of a body, one has

$$f = \frac{Gm_gM}{r^2} = m_Ia, \quad \text{or} \quad a = \frac{m_g}{m_I}\left(\frac{GM}{r^2}\right)$$

which is constant if m_g/m_I is a constant.] Thus the equality of inertial and gravitational mass follows from the principle of equivalence. It was shown by Eötvös that the inertial and gravitational masses of a body are equal to within 1 part in 10^9. For a recent discussion the reader is referred to Dicke[2]. It should be remembered that in these experiments, the particles had low velocities, so that the equality of gravitational mass and inertial mass has not been checked at high velocities comparable to the velocity of light.

In a region where the gravitational field is uniform one can 'transform away' the gravitational field by transforming to a co-ordinate accelerating in the direction of the gravitational field with an acceleration numerically equal to the acceleration due to the gravitational field. The new co-ordinate system is then an inertial reference frame. In general, gravitational fields are not uniform (or homogeneous) throughout all space, so that one cannot replace the gravitational fields, throughout all space by transforming to a single co-ordinate system accelerating relative to the first. For example, one cannot 'transform away' the gravitational field of the earth throughout all space.

To quote Fock[3]:

> The Principle of Equivalence is related to the fundamental law of equality of inertial and gravitational mass, but it is not identical with that law. The latter is a law of general, not of local, character whereas the equivalence of accelerations and gravitational fields is entirely local, i.e. refers to a single point in space (more exactly, to the spatial neighbourhood of the points on a time-like world line).
>
> The equivalence amounts to the following. By introducing a suitable system of co-ordinates (which is usually interpreted as an accelerated frame of reference) one can so transform the equations of motion of a mass point in a gravitational field that in this new system they will have the appearance of equations of motion of a *free* mass point. Thus a gravitational field can, so to speak, be replaced, or rather imitated by a field of acceleration. Owing to the equality of inertial and gravitational

mass such a transformation is the same for any value of the mass of the particle. But it will succeed in its purpose only in an infinitesimal region of space, i.e. it will be strictly local.

9.3. THE RATES OF CLOCKS IN GRAVITATIONAL FIELDS

The principle of equivalence will now be applied to calculate the rates of clocks in a gravitational field. It will be assumed that all velocities are very much less than the velocity of light, and we shall work to second order of v/c only.

Consider a rocket of length h moving with acceleration $\mathbf{a}$ relative to an inertial frame Σ, in which there is no gravitational field, as shown in *Figure* 9.2(a). Let the rocket be instantaneously at rest in Σ at the time $t = 0$. Let two clocks of identical construction, labelled 1 and 2, be fixed to the rocket, with clock 1 at the rear end and clock 2 at the front end of the rocket, as shown in *Figure* 9.2(a). As examples of suitable clocks, two light sources are chosen which have a frequency ν_0 when the rocket is at rest in an inertial reference frame in which there is no gravitational field. Consider the light emitted from clock 1 at a time $t = 0$, when the rocket is instantaneously at rest in the inertial reference frame Σ. This light reaches clock 2 after a time interval t (measured in Σ) given by $ct = (h + \frac{1}{2}at^2)$, which reduces to $t = h/c$ if $\frac{1}{2}at \ll c$. After a time t, clock 2 is moving with a velocity $v = at \simeq ah/c$ relative to Σ. To an observer situated by clock 2 it will appear as if the light reaching him from clock 1 was emitted by a source moving downwards with velocity v, so that he will observe the light to be Doppler shifted, the frequency being, to first order, $\nu' = \nu_0(1 - v/c) = \nu_0(1 - ah/c^2)$. This is a lower frequency than that of the light from clock 2. Light emitted from clock 2 at the instant $t = 0$ in Σ, has a frequency $\nu_0(1 + ah/c^2)$ relative to clock 1 when it reaches clock 1. To first order v/c the difference in both cases is $\nu_0 ah/c^2$, if $ah/c^2 \ll 1$.

According to the principle of equivalence, these experimental results should be exactly the same as when the same experiments are carried out in the non-accelerating co-ordinate system K, shown in *Figure* 9.2(c), in which the rocket is at rest and in which there is a gravitational field of such a strength that g, the acceleration due to the gravitational field, is numerically equal, but opposite in direction, to $\mathbf{a}$, the acceleration of the rocket relative to Σ. The light reaching clock 2 from clock 1 should have a frequency lower than clock 2 by an amount $\nu_0 gh/c^2$. Conversely, light reaching clock 1 from clock 2 should have a frequency higher than the

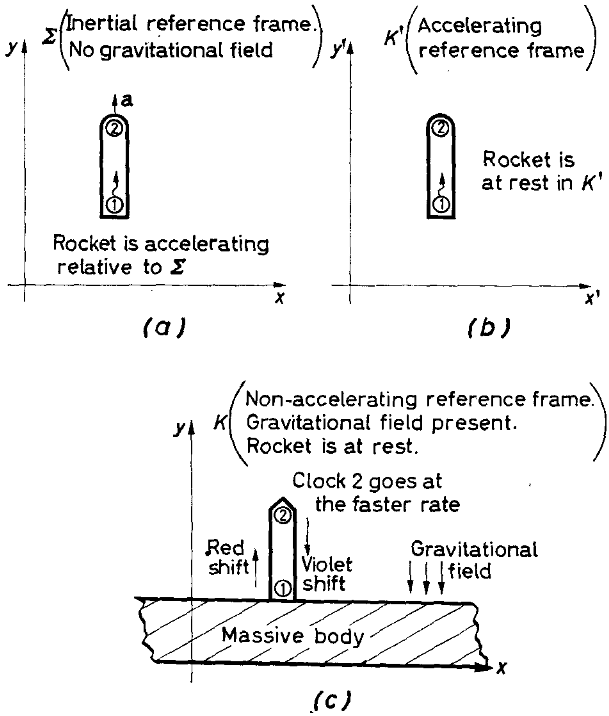

Figure 9.2. (a) In the inertial reference frame Σ the rocket has an acceleration **a**. *Light is emitted from two identical light sources* 1 *and* 2 *at the time $t = 0$, when the rocket is instantaneously at rest in Σ. By the time light from source* 1 *reaches source* 2, *source* 2 *is moving relative to Σ, so that the light from source* 1 *will appear Doppler shifted to an observer by source* 2. *(b) In the accelerating reference frame K', in which the rocket is at rest, the light reaching source* 2 *from source* 1 *has a lower frequency than source* 2. *According to the principle of equivalence, measurements in K' should be the same as in the non-accelerating reference frame K, shown in (c), in which there is a gravitational field. Hence clock* 2 *in (c) should go at a faster rate than* 1 *in K, and visible light from light source* 1 *should be shifted to the red end of the spectrum compared with the frequency of the exactly similar light source* 2

frequency of clock 1. The light sources are both at rest in K. No light vibrations should be lost on the way, so that one must conclude that, in the presence of a gravitational field, the actual frequencies of the stationary clocks differ by an amount $\nu_0 gh/c^2$. This is an actual physical difference associated with the presence of a gravitational field. Now gh is the difference in gravitational potential between the clocks. According to normal convention,

clock 2 is at the higher gravitational potential in K [cf. *Figure 9.2(c)*], since work must be done to raise a mass from clock 1 to clock 2 in the gravitational field in K. So far, it has been assumed that g is constant between the two clocks. For a varying gravitational field, gh has to be replaced by $\Delta\phi$, where $\Delta\phi$ is the difference in gravitational potential between the clocks, so that $\Delta\nu$, the difference between the frequencies, is $\nu_0\Delta\phi/c^2$. Hence, the fractional difference in frequency is

$$\frac{\Delta\nu}{\nu_0} \simeq \frac{\Delta\phi}{c^2} \tag{9.1}$$

If ν_1 is the frequency of clock 1, and ν_2 is the frequency of clock 2 when the clocks are at rest in a gravitational field [e.g. as in *Figure 9.2(c)*] provided $\Delta\phi \ll c^2$,

$$\nu_2 = \nu_1\left(1 + \frac{\Delta\phi}{c^2}\right) \tag{9.2}$$

and

$$\nu_1 = \nu_2\left(1 - \frac{\Delta\phi}{c^2}\right) \tag{9.3}$$

Eqn (9.2) should hold for other processes, such as the successive ticks of clocks. According to eqn (9.2), clock 2 should register more ticks in a given time interval, and so should register a longer time interval than clock 1. If Δt_1 and Δt_2 are the time intervals registered by clocks 1 and 2 respectively, since the time interval measured by each clock is proportional to its frequency, from eqn (9.2),

$$\Delta t_2 = \Delta t_1\left(1 + \frac{\Delta\phi}{c^2}\right) \tag{9.4}$$

or

$$\Delta t_1 = \Delta t_2\left(1 - \frac{\Delta\phi}{c^2}\right) \tag{9.5}$$

where $\Delta\phi$ is the difference in gravitational potential between clocks 1 and 2. Commenting on eqn (9.4), Born[4] writes:

> The physical content of this formula is this: Given two equally constructed, synchronous clocks initially at rest relative to each other; if one of them is exposed to a gravitational field for a certain period of time, they will no longer be synchronized, but the clock which was in the field will be retarded.

The reader should remember that, in the above quotation, the clock in the gravitational field is at a lower gravitational potential

than the clock in the field-free region, so that the clock in the field corresponds to clock 1 in eqn (9.4), and according to eqn (9.4) clock 1 should go at the slower rate. All processes should take place faster near clock 2 on the top of the stationary rocket [illustrated in *Figure 9.2(c)*] than at the bottom of the rocket near clock 1. For example, radioactive atoms should decay faster near clock 2 than near clock 1. The changes are illustrated in *Figure 9.2(c)* where it is assumed that the gravitational field arises from a massive body. At the higher altitude everything goes faster and the frequency of light reaching the lower altitude is higher than the frequency of light from an exactly similar light source at the lower altitude. For visible light, light coming from the higher altitude (or higher gravitational potential) is shifted to the violet end of the spectrum. One has a red shift for light going from the lower to the higher altitude (i.e. from lower to higher gravitational potential).

9.4. THE GRAVITATIONAL SHIFT OF SPECTRAL LINES

It was shown in Section 9.3 that it follows from the principle of equivalence that the frequency of a light source is influenced by a gravitational field. Let a light source of frequency ν_1, near a star correspond to clock 1 and let a similar light source on the earth have frequency ν_2, and correspond to clock 2. According to eqn (9.3),

$$\nu_1 = \nu_2\left(1 - \frac{\Delta\phi}{c^2}\right) \tag{9.3}$$

The difference in gravitational potential $\Delta\phi$, between clocks 1 and 2 is positive, since net work would have to be done to take a mass from the more massive star to the earth. Hence ν_1 is less than ν_2, so that $\lambda_1 > \lambda_2$, that is, visible light coming from a star is shifted towards the red end of the spectrum compared with light from an exactly similar light source stationary on the earth. If $\Delta\lambda$ is the increase in wavelength,

$$\frac{\Delta\lambda}{\lambda} \simeq \frac{\Delta\phi}{c^2} \tag{9.6}$$

For light coming from the sun the fractional shift in a spectral line is $\Delta\lambda/\lambda \sim 2{\cdot}12 \times 10^{-6}$ compared with the wavelength of the same spectral line from a stationary terrestrial source. This is a measurable effect, but it is equivalent to the ordinary Doppler effect for a source moving with a velocity of only 0·63 km/sec, so

that the gravitational red shift of light from the sun may be masked by other effects. According to Dicke[2], Brault has measured the red shift of the solar sodium D_1 line. To quote Dicke[2], 'After correction for asymmetrics, the red shift was found to agree with the expected result with an accuracy of 5 per cent (noise limited)'. White dwarf stars are more massive than the sun, so that $\Delta\phi$ and hence $\Delta\lambda$ should be correspondingly larger. Though not definitive, the results obtained with light from white dwarfs are consistent with eqn (9.6).

At one time it was felt that it would be impossible to check the gravitational shift of spectral lines by means of terrestrial experiments. For example, the change in frequency between sea level and the top of Mount Everest is only about 1 part in 10^{12}. However, atomic clocks are beginning to approach the required accuracy. With the discovery of the Mössbauer effect, it became possible to measure extremely small changes in frequency. All that was necessary was to mount the source and the absorber vertically above each other at sea level. The source and absorber would correspond to clocks 1 and 2 in *Figure 9.2(c)*. Any shift in characteristic frequency between source and absorber would change the proportion of recoilless absorptions leading to a change in the counting rate of the detector (cf. *Figure A5.4*). In the experiment carried out by Pound and Rebka[5], a 0·4 curie source of ^{57}Co in metallic iron was separated by a vertical height of 22 m from the absorber. Pound and Rebka controlled the temperature very accurately, since a temperature difference of 1°K between source and absorber would produce an effect comparable with the gravitational shift of the 14·4 keV line. (The temperature dependence of the Mössbauer effect was discussed in Section 8.4.) The shift expected by Pound and Rebka was about one part in $2{\cdot}4 \times 10^{-15}$. They verified the existence of the gravitational shift and showed that it was $0{\cdot}97 \pm 0{\cdot}04$ times the predicted value.

It can be seen that one need not use the full theory of general relativity to calculate the gravitational shift of spectral lines. The shift can be calculated sufficiently accurately using the principle of equivalence only.

9.5. THE CLOCK PARADOX INTERPRETED IN TERMS OF THE PRINCIPLE OF EQUIVALENCE

The example given in Section 8.1 and illustrated in *Figure 8.1* is now reconsidered. In this example, a rocket leaves the earth with uniform velocity v, turns around quickly and returns with

uniform velocity v to the earth. If t_L is the time of the complete journey measured by a clock at rest on the earth, and if t'_R is the time for the complete journey measured by a clock at rest on the rocket, then, if the clock hypothesis [eqn (8.8)] is assumed to be correct and is applied in the inertial frame in which the earth is at rest, it is found that

$$t'_R = t_L\sqrt{(1 - v^2/c^2)} \simeq t_L(1 - \tfrac{1}{2}v^2/c^2) \tag{8.6}$$

The motion is now considered from the reference frames in which the rocket is instantaneously at rest. For the first part of the journey, when the rocket is moving with uniform velocity v relative to the earth, the rocket is at rest in an inertial frame, which is denoted by K_1, as shown in *Figure 9.3(a)*. The earth is moving to the left with uniform velocity v in K_1. Let the time interval between the earth's leaving the rocket and the instant the earth starts to turn around be t'_1 as measured by clocks at rest relative to the rocket, and t_1 as measured by a clock at rest on the earth. In K_1, t_1 is a proper time interval so that, according to the theory of special relativity,

$$t'_1 = \frac{t_1}{\sqrt{(1 - v^2/c^2)}} \simeq t_1(1 + \tfrac{1}{2}v^2/c^2) \tag{9.7}$$

When the rocket is turning around relative to the earth it has an acceleration $a = 2v/t_2$ relative to the earth, where t_2 is the time (relative to the earth) taken by the rocket to turn around and regain a velocity v in the opposite direction. The reference frame in which the rocket is now at rest is an accelerating reference frame. According to the principle of equivalence, the results in this accelerating frame would be exactly the same as if the measurements were carried out in a 'stationary' co-ordinate system K_2 in which the rocket is at rest, and in which there is a homogeneous gravitational field, in which the acceleration due to gravity is numerically equal to the acceleration of the rocket relative to the inertial frame in which the earth is at rest, that is, $g = 2v/t_2$, as illustrated in *Figure 9.3(b)*. In the reference frame K_2, the earth does the equivalent of 'vertical' motion under gravity. In K_2, the earth is in a region of higher gravitational potential than the rocket. Relative to K_2, all clocks on the earth should go at a faster rate than similar clocks on the rocket. If t_2 is the time for the earth to turn around (relative to clocks at rest on the earth) and t'_2 is the same time interval measured relative to the rocket, according to eqn (9.5),

$$t'_2 = t_2\left(1 - \frac{gh}{c^2}\right) \tag{9.8}$$

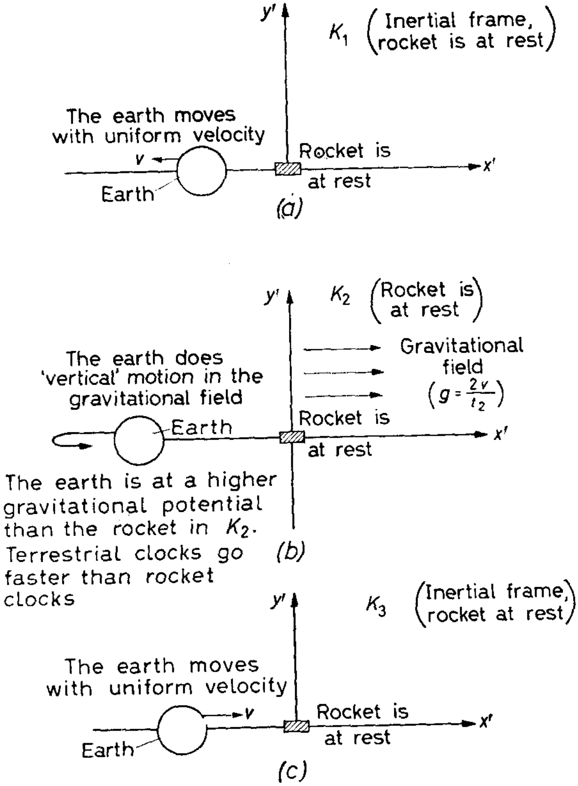

Figure 9.3. An example of the clock paradox. The motion is considered from the co-ordinate system in which the rocket is always at rest. The journey is considered in three parts. In the first part, illustrated in (a), the earth moves with uniform velocity away from the rocket, which is at rest in the inertial frame K_1 during this part of the journey. In the second part of the journey, shown in (b), the earth is accelerating relative to the rocket, and relative to the rocket the earth appears to be moving in a gravitational field. During the last stage of the journey (c) the earth is moving with uniform velocity towards the rocket, which is at rest in the inertial reference frame K_3

where $h = vt_1'$ is the distance of the earth from the rocket measured relative to the rocket, and $g = 2v/t_2$. Substituting for t_1' from eqn (9.7), one finds for h,

$$h = vt_1' = vt_1(1 + \tfrac{1}{2}v^2/c^2)$$

Substituting for g and h in eqn (9.8),

$$t_2' = t_2\left[1 - \frac{2v\, vt_1}{t_2\, c^2}(1 + \tfrac{1}{2}v^2/c^2)\right]$$

or, to second order,

$$t_2' = t_2 - \frac{2v^2}{c^2} t_1 \tag{9.9}$$

Note, the second term in eqn (9.9) is proportional to t_1. Even if t_2 is very much less than t_1, so that t_2 can be neglected when calculating t_L (as was done in Section 8.1), the time interval t_2' cannot be neglected when the motion is considered relative to the reference frame in which the rocket is always at rest. If the acceleration of the rocket relative to the earth is increased, so that t_2 is smaller, g is increased and the second term in eqn (9.9) remains precisely the same. If the length of the journey is increased, then relative to K_2 the earth is in a region of still higher gravitational potential when the earth turns around, the difference in potential being proportional to vt_1. [If the rocket accelerates from rest at the beginning of the journey, the difference in gravitational potential between the earth and the rocket is practically zero, and the acceleration time is (to second order) the same relative to the earth and the rocket and can be made negligible compared with t_1 and t_1'. A similar argument can be applied if the rocket comes to rest at the end of the journey.]

Once the rocket has stopped accelerating with respect to the earth, and is moving with uniform velocity v again, the rocket is at rest in the inertial reference frame K_3 shown in *Figure 9.3(c)*. The theory of special relativity is applicable again, so that, if t_3 and t_3' are the times for the return journey of the earth to the rocket measured by a clock on the earth and by clocks at rest relative to the rocket respectively, then since relative to K_3, t_3 is the proper time interval,

$$t_3' = t_3(1 - v^2/c^2)^{-\frac{1}{2}} \simeq t_3(1 + \tfrac{1}{2}v^2/c^2) \tag{9.10}$$

Adding eqns (9.7), (9.9) and (9.10),

$$t_1' + t_2' + t_3' = t_1 + t_2 + t_3 + t_1[(\tfrac{1}{2}v^2/c^2) - (2v^2/c^2)] + t_3\tfrac{1}{2}v^2/c^2$$

Since relative to the earth t_2 is negligible, then $t_1 = t_3 = t_L/2$, so that

$$t_R' = t_L(1 - \tfrac{1}{2}v^2/c^2) \tag{9.11}$$

Eqn (9.11) is the same (to second order of v^2/c^2) as the result obtained previously in Section 8.1 [cf. eqn (8.6)], when the

motion was always related to the inertial frame in which the earth was at rest.

It should be emphasized that we cannot claim to have resolved the clock paradox. What has been done is to work out the problem in a different co-ordinate system to the one used in Section 8.1 and the same answer was obtained. This is similar to the case of the motion of a body on a rotating roundabout, when one obtains the same answer if one uses a rotating co-ordinate system or refers everything to the lab.-system (cf. Section 1.5).

9.6. THE RATES OF CLOCKS IN SATELLITES

With the advent of earth satellites, interest in the clock paradox has been stimulated again recently. When a satellite is in orbit above the earth there are two effects to be considered when comparing the rates of clocks on the earth and in the satellite. Firstly, if one accepts the clock hypothesis [eqn (8.8)], there is an effect due to the motion of the satellite relative to the earth. Let ν_0 be the frequency of a process measured on the earth, and let $\nu_0 + \Delta\nu_1$ be the frequency of the same process measured in a satellite moving with uniform speed v. From eqn (8.8),

$$t'_R = t_L\sqrt{(1 - v^2/c^2)} \simeq t_L(1 - \tfrac{1}{2}v^2/c^2)$$

Since ν_0 is proportional to t_L and $\nu_0 + \Delta\nu_1$ is proportional to t'_R, the increase in the frequency of a clock in the satellite is given, to second order, by

$$\frac{\Delta\nu_1}{\nu_0} = -\tfrac{1}{2}v^2/c^2 \tag{9.12}$$

that is, the clock in the satellite goes at a slower rate than a similar clock stationary on the earth. If m is the mass of the satellite, M the mass of the earth, G the gravitational constant, r_0 the radius of the earth, and r the geocentric radius of the satellite's orbit (which is assumed to be circular), one has

$$\frac{mv^2}{r} = \frac{mGM}{r^2}$$

or

$$v^2 = \frac{GM}{r} = \frac{GM}{r_0}\left\{\frac{r_0}{r}\right\} = gr_0\left\{\frac{r_0}{r}\right\} \tag{9.13}$$

In eqn (9.13), g, which is equal to GM/r_0^2, is the acceleration due to gravity at the surface of the earth. Substituting in eqn (9.12),

$$\frac{\Delta\nu_1}{\nu_0} = -\frac{1}{2}\frac{gr_0}{c^2}\left(\frac{r_0}{r}\right) \tag{9.14}$$

Secondly, the clock in the satellite is at a higher gravitational potential than a similar clock on the earth, and consequently should go at a faster rate. According to eqn (9.1) the increase in frequency $\Delta\nu_2$ of the clock in the satellite due to the difference in gravitational potential is given by

$$\frac{\Delta\nu_2}{\nu_0} = \frac{\Delta\phi}{c^2}$$

But,

$$\Delta\phi = \int_{r_0}^{r} \frac{GM}{r^2}\,dr = -GM\left(\frac{1}{r} - \frac{1}{r_0}\right)$$

$$= \frac{GM}{r_0}\left(1 - \frac{r_0}{r}\right) = gr_0\left(1 - \frac{r_0}{r}\right)$$

Hence,

$$\frac{\Delta\nu_2}{\nu_0} = \frac{gr_0}{c^2}\left(1 - \frac{r_0}{r}\right) \tag{9.15}$$

Adding eqns (9.14) and (9.15), one finds for $\Delta\nu$, the total increase in the frequency of the clock in the satellite,

$$\frac{\Delta\nu}{\nu_0} = \frac{gr_0}{c^2}\left\{1 - \frac{3r_0}{2r}\right\} \tag{9.16}$$

If $\frac{3r_0}{2r} < 1$ or $r > \frac{3}{2}r_0$, that is, if the altitude of the satellite is $> \frac{r_0}{2}$ ($\simeq$3,200 km above sea level), $\Delta\nu$ is positive, since the effect due to the change in gravitational potential predominates, and a clock in the satellite reads more (and the astronaut ages more) than a similar clock (or an astronaut) at rest on the earth. For $r < 3r_0/2$, that is, if the altitude of the satellite is less than $r_0/2 \simeq 3{,}200$ km above sea level, then the effect given by eqn (9.14) predominates, and the clock on the satellite reads less than a clock on the earth, and the astronaut should age less than a person on the earth.

We are now in a position to understand the quotation taken from Cochran[6] and given in Section 8.5. The clock in the rocket projected vertically is in a region of higher gravitational potential than a clock at rest on the platform and consequently goes at a faster rate. The clock in a low altitude satellite goes slower than the clock at rest on the platform as it is moving relative to the clock at rest on the platform in a region where the gravitational potential is the same as on the platform.

9.7. THE BENDING OF LIGHT IN A GRAVITATIONAL FIELD

In Newtonian mechanics one could in principle use rigid bodies as reference bodies and rulers to measure co-ordinates of events. The classical concept of rigid bodies had to be refined in the theory of special relativity, but one could still in principle use light signals, *in vacuo*, to define straight lines and measure the times and co-ordinates of events, since according to the theory of special relativity light should travel in straight lines with uniform speed *in vacuo* relative to inertial reference frames.

Consider a space-ship moving with uniform acceleration relative to an inertial reference frame Σ. If light enters through a small hole in the space-ship, it travels in a straight line relative to Σ. Relative to the accelerating space-ship the light must travel in a curved path. According to the principle of equivalence, observations on the accelerating space-ship should be the same as observations carried out by a stationary observer in a homogeneous gravitational field, of such a strength that the acceleration due to gravity in the gravitational field is equal and opposite to the acceleration of the space-ship. Hence, light rays should be curved in a gravitational field, the light rays being bent in the direction of the acceleration due to gravity.

As an example, Einstein considered the deflection of light rays from a distant star, when the light rays passed close to the periphery of the sun. Using the difference in clock periods calculated using the principle of equivalence, and given by eqn (9.1), Einstein showed that a light ray which passes the sun at a distance R from its centre should be deflected through an angle α given by

$$\alpha = \frac{2GM}{c^2R} \tag{9.17}$$

where G is the gravitational constant and M is the mass of the sun.

Substituting for G, M and the value of the radius of the sun for R, it follows from eqn (9.17), that $\alpha = 0{\cdot}87''$ of arc.

There is an alternative approach based on the equality of inertial and gravitational mass. It was shown in Section 5.8.3, that a photon of energy $h\nu$ has associated with it an inertial mass $h\nu/c^2$, so that the photon should have associated with it a gravitational mass $h\nu/c^2$ and should be attracted towards the sun. If one considers a photon incident at an impact parameter R, a straightforward application of the theory of collisions shows that the deflection of the photon is again given by eqn (9.17) (cf. French[7]).

After developing his full theory, Einstein recalculated the deflection of light passing near the sun using the full theory of general relativity, and he found that the deflection of light by the sun should be double that given by eqn (9.17), that is, $1{\cdot}75''$ of arc. It has been shown by Schiff[8] that the value predicted by the full theory of general relativity follows from the principle of equivalence if one allows for both the changes in the rates of clocks, and in the lengths of rods.

The light rays from stars which pass close to the sun are generally invisible, except during a solar eclipse. Two British expeditions investigated the deflection of light by the sun during the total eclipse of 29th May, 1919. One group went to the Isle of Principe in the Gulf of Guinea, West Africa and the other went to Sobral in North Brazil. After correcting for systematic errors the final results (reduced to the edge of the sun) obtained at Sobral and Principe with their 'probable accidental errors', were

Sobral	$1''{\cdot}98 \pm 0''{\cdot}12$
Principe	$1''{\cdot}61 \pm 0''{\cdot}30$

These were consistent with the value calculated on the basis of the full theory of general relativity, which is double that given by eqn (9.17). A fuller account of the observations is given by Eddington[9]. The results have been confirmed by experiments carried out during later eclipses of the sun.

It has been shown in this section that, in the strict Euclidean sense, light does not travel in straight lines in a gravitational field, so that light rays cannot be used to mark out a rectangular Cartesian co-ordinate system in which Euclidean geometry can be applied. It has been shown that the rates of clocks depend on the gravitational potential. Lengths of rods also depend on the strength of the gravitational field. It is, therefore, not surprising to find that, in the full theory of general relativity, the numerical value of the speed of light depends on the strength of the gravitational field. Hence

the principle of the constancy of the velocity of light must be modified in the presence of strong gravitational fields. To quote Einstein[10]:

> · · · our result shows that, according to the general theory of relativity, the law of the constancy of the velocity of light *in vacuo*, which constitutes one of the two assumptions in the special theory of relativity and to which we have already frequently referred, cannot claim any unlimited validity. A curvature of rays of light can only take place when the velocity of propagation of light varies with position. Now we might think that as a consequence of this, the special theory of relativity, and with it the whole theory of relativity, would be laid in the dust. But in reality this is not the case. We can only conclude that the special theory of relativity cannot claim an unlimited domain of validity; its results hold only so long as we are able to disregard the influences of gravitational fields on the phenomena (e.g. of light).

Near the earth the gravitational field is so weak that the deviations of light rays from straight lines can, for all practical purposes, be neglected, though the reader should bear in mind that there are limitations to the principle of the constancy of the velocity of light.

9.8. ROTATING REFERENCE FRAMES

It was pointed out in Section 1.5 that if one wants to apply Newton's laws of motion in a reference frame rotating with uniform angular velocity $\boldsymbol{\omega}$ relative to an inertial frame, one has to introduce the centrifugal force $-m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$ and the Coriolis force $-2m\boldsymbol{\omega} \times \mathbf{u}'$. In Section 1.5 these forces are called fictitious, as they produce no acceleration relative to an inertial frame. It is interesting to note that both the centrifugal and Coriolis forces acting on a body are proportional to the mass of the body. Newton felt that absolute rotations could be measured in the way described in Section 1.5. For example, Newton suggested that from the tension in the cord joining two globes rotating about their common centre of gravity, one can determine the amount of their rotation. Newton took this to be evidence in favour of absolute space. This view was criticized some twenty years later by Bishop Berkeley, who wrote:

> If every place is relative, then every motion is relative and as motion cannot be understood without a determination of its direction which in its turn cannot be understood except in relation to our or some other body. Up, down, right, left, all directions and places are based on some relation and it is necessary to suppose another body distant from the moving one · · · so that motion is relative in its nature, it cannot be understood

until the bodies are given in relation to which it exists, or generally there cannot be any relation, if there are no terms to be related. Therefore, if we suppose that everything is annihilated except one globe, it would be impossible to imagine any movement of that globe. Let us imagine two globes and that besides them nothing else material exists, then the motion in a circle of these globes round their common centre cannot be imagined. But suppose the heaven of fixed stars were suddenly created and we shall be in a position to imagine the motion of the globes by their relative position to the different parts of the heaven.

Newton's views were also criticized by Mach in the second half of the nineteenth century; e.g. Mach wrote:

Obviously it does not matter if we think of the earth as turning round on its axis, or at rest while the fixed stars revolve round it. Geometrically these are exactly the same case of a relative rotation of the earth and the fixed stars with respect to one another. But if we think of the earth at rest and the fixed stars revolving round it, there is no flattening of the earth, no Foucault's experiment, and so on—at least according to our usual conception of the law of inertia. Now one can solve the difficulty in two ways. Either all motion is absolute, or our law of inertia is wrongly expressed. I prefer the second way. The law of inertia must be so conceived that exactly the same thing results from the second supposition as from the first. By this it will be evident that in its expression, regard must be paid to the masses of the universe.

The quotations from Berkeley and Mach are taken from Sciama[11]. Mach points out that one cannot conclude from the flattening of the earth, due to its rotation about its axis, that absolute space exists. All one can conclude is that the effect is associated with the rotation of the earth relative to the rest of the matter in the universe. Mach felt that, if the theory were properly expressed, it should give the same results when the earth is rotating relative to the rest of the matter in the universe, as would be obtained if the rest of the matter in the universe were to rotate around a stationary earth. If the laws of physics are to be the same in all reference frames, as required by the principle of covariance, then the centrifugal and Coriolis forces must come naturally out of the general theory of relativity.

The interpretation of phenomena in a rotating reference frame, suggested by Mach, is illustrated by a simple example. Consider a circular roundabout rotating inside a concentric circular room. An observer on the rotating roundabout will tend to continue in a straight line relative to the fixed stars, and if he sits on the roundabout he will feel himself tending to fly outwards. If the roundabout stopped rotating and the circular wall started rotating in the opposite direction, when he looked at the wall the observer would see the

same things on the circular wall going in the same direction as before, but he would no longer feel forces tending to push him outwards. He could carry out experiments to 'show' that the roundabout was rotating in the first instance but not in the second. For example, he could attach a weight by a spring balance to the axis of the roundabout. If the weight rested on the roundabout, the spring balance would register a force in the first instance but not in the second. However, the two systems considered so far are not really equivalent. In the first case not only is the roundabout rotating relative to the walls of the room, but it is rotating relative to the rest of the matter in the universe. To be completely equivalent all the other masses in the universe would have to rotate around the stationary roundabout in the second case. If they did so, would the 'stationary' observer then feel a force tending to pull him outwards from the centre of the stationary roundabout? The generally accepted view, at present, is that he would. The origin of such a force is generally attributed to gravitational fields due to the motion of the distant masses in the universe.

According to Newton's theory of gravitation, the force of gravitational attraction between two masses m_1 and m_2 is equal to $G(m_1 m_2)/r^2$, whether the masses are moving or not. According to the theory of special relativity, force is not absolute [cf. eqns (5.74), (5.75) and (5.76)]. Hence, if the two masses are moving relative to an observer, the force of gravitational attraction between them should be measured to be different from the value obtained when the masses are at rest relative to the observer. As an analogy, consider two electric charges. When the charges are at rest relative to an inertial frame, the electrostatic force between them is given by Coulomb's law. If the charges are moving, then there are velocity dependent magnetic forces between the charges in addition to electric forces. If the electric charges are accelerating there is a contribution to the electric and magnetic fields which depends on the acceleration of the charges. The acceleration dependent fields are the radiation terms, which vary as $1/r$. The radiation terms are not symmetrical in the direction of the acceleration of a charge and the direction opposite to it, except in the limit of zero velocity. Similarly, if masses are accelerating relative to the observer it is quite feasible that the gravitational forces between two masses depends on the acceleration of the masses relative to the reference frame chosen. Thus it is feasible that the masses of the universe rotating around a stationary roundabout would give rise to gravitational forces tending to pull the observer away from the centre of the roundabout. In the case of the stationary roundabout, the centrifugal force

would be interpreted as a gravitational force arising from the rotation of the rest of the masses of the universe around the stationary roundabout. It is well known that the centrifugal force due to the rotation of the earth and the gravitational force due to the mass of the earth cannot be separated experimentally, and for this reason tables of the acceleration due to gravity at various points on the earth's surface often include the contribution of the centrifugal force due to the rotation of the earth. (One can, of course, estimate the contribution due to the centrifugal force, but one cannot separate it experimentally from the gravitational force due to the earth.)

Because of their great distances from the earth it might appear, at first sight, impossible for distant stars to affect terrestrial phenomena. The gravitational field due to a stationary mass is proportional to $1/r^2$, whilst the contribution to the gravitational field due to the acceleration of a mass may very well be proportional to $1/r$ (as is the case in electromagnetism). These fields do not necessarily cancel to zero for points off the axis of the roundabout. The number of stars between r and $r + \mathrm{d}r$ is proportional to $r^2\,\mathrm{d}r$, so that for a $1/r$ law the contribution to the field near the centre of the stationary roundabout would be proportional to $r\,\mathrm{d}r$. Hence the total force would diverge, if one could integrate from 0 to ∞. This would be true for a $1/r^2$ law also. Of course, the actual total gravitational force would depend on the boundary conditions, such as the size of the universe, the recession of distant nebulae, etc. However, the possibility exists that the distant stars may have some effect on terrestrial phenomena. According to the principle of covariance one should not be able to say by experiment whether the roundabout is rotating relative to the rest of the matter in the universe, or whether the rest of the matter in the universe is rotating relative to the roundabout; the laws of physics should be the same in both cases.

We shall now consider the rate of a clock which is fixed to a roundabout which is rotating with angular velocity ω relative to an inertial frame Σ. Let the clock be at a distance r from the axis of rotation of the roundabout. The theory of special relativity can be applied in the inertial frame Σ. The clock is moving with a linear velocity $r\omega$ relative to Σ. If ν is the frequency of a clock stationary in Σ (or a clock on the axis of rotation of the roundabout) and ν_0 is the frequency of a clock on the roundabout, using the clock hypothesis [eqn (8.8)] and the theory of special relativity,

$$\nu_0 = \nu\sqrt{(1 - v^2/c^2)} \simeq \nu\left(1 - \tfrac{1}{2}r^2\frac{\omega^2}{c^2}\right) \tag{9.18}$$

The rates of the clocks will now be considered from the reference frame in which the roundabout is at rest, and in which the rest of the masses in the universe are rotating. It is fairly generally accepted that in this reference frame there should be a gravitational field (generally attributed to the rotation of the stars), which gives the correct centrifugal and Coriolis forces. To quote Møller[12]:

> There is one effect of this kind, however, which, although small, is of theoretical importance since it throws new light on the origin and nature of the centrifugal and Coriolis forces appearing in a rotating system of co-ordinates S. According to the idea of Einstein underlying the general principle of relativity (cf. Section 82), these forces are gravitational forces originating from the rotation of the distant celestial masses relative to S, and such 'non-permanent' gravitational fields should satisfy the same general field equations as the permanent fields. The approximate solutions (34) for weak fields do not directly allow us to treat the effects of the distant celestial masses, but we may expect that a rotating spherical shell of uniform mass density will produce effects inside the shell similar to the rotation of the distant celestial masses $\cdots$.
>
> For a rotating shell of matter, Thirring, [*Phys. Z.* **19**, (1918) 33; **22** (1921) 29)] found the interesting result that the field in the interior of the shell as determined by the equations (30), is similar to the field in a rotating system of co-ordinates thus leading to gravitational forces similar to the usual centrifugal and Coriolis forces.

For a complete discussion the reader is referred to Møller[12] and to Thirring. In order to illustrate the interpretation of phenomena on a rotating roundabout, we shall consider a simplified model. The centrifugal force acting on a body of mass m resting at a distance r from the axis of a rotating roundabout is $mr\omega^2$. It acts away from the axis of rotation. The work done in taking a unit mass at negligible speed from the axis to a radial distance r is $-\int_0^r r\omega^2\,dr = -\frac{1}{2}r^2\omega^2$. If the centrifugal force is a gravitational force, relative to the roundabout, the scalar gravitational potential difference due to the centrifugal gravitational field is equal to

$$\Delta\phi = -\tfrac{1}{2}r^2\omega^2 \tag{9.19}$$

When calculating the above scalar gravitational potential difference, effects associated with the velocity dependent Coriolis forces were ignored. This is analogous to neglecting the magnetic force on an electric charge moved from one point to another. It will be assumed that a clock on the axis of the roundabout (whose linear velocity relative to the laboratory reference frame Σ is zero), is synchronous with all the clocks at rest in Σ. If the difference in gravitational

potential between a clock of frequency ν on the axis of the roundabout and a clock of frequency ν_0 at a radial distance r is given by eqn (9.19), then, according to eqn (9.2), the clock on the axis should go at the faster rate, since it is at the higher gravitational potential. From eqn (9.2) [cf. *Figure 9.2(c)*],

$$\nu_0 = \nu\left(1 + \frac{\Delta\phi}{c^2}\right) = \nu\left(1 - \frac{r^2\omega^2}{2c^2}\right) \tag{9.20}$$

This is in agreement with eqn (9.18). An experiment of this type was carried out by Hay, Schiffer, Cranshaw and Egelstaff[13] using the Mössbauer effect. The experiment is outlined in Section 8.4. Their results were in agreement with eqns (9.18) and (9.20).

The agreement between eqns (9.18) and (9.20) shows that one gets the same answer whether one uses the theory of special relativity in an inertial reference frame, or if one uses a non-inertial rotating reference frame, provided one introduces the centrifugal gravitational field in the rotating system. To quote Born[14]:

> Thus from Einstein's point of view Ptolemy and Copernicus are equally right. What point of view is chosen is a matter of expediency. For the mechanics of the planetary system the view of Copernicus is certainly the more convenient. But it is meaningless to call the gravitational fields that occur when a different system of reference is chosen 'fictitious' in contrast with the 'real' fields produced by near masses: it is just as meaningless as the question of the 'real' length of a rod in the special theory of relativity. A gravitational field is neither 'real' nor 'fictitious' in itself. If has no meaning at all independent of the choice of co-ordinates just as in the case of the length of a rod. Nor are the fields distinguished by the fact that some are directly produced by masses while others are not; in the one case it particularly is the near masses that produce an effect; in the other it is the distant masses of the cosmos.

In addition to the rates of clocks, the lengths of rods are affected by rotation. Let the rotating roundabout, shown in *Figure 9.4*, rotate with angular velocity ω relative to Σ. Let an observer O' be at rest on the rotating roundabout. Let the observer O' have a measuring rod of proper length l_0, when it is at rest in an inertial reference frame. The observer O' sets out to measure the diameter and the circumference of the rotating roundabout. His operations are considered from the inertial frame Σ in which the roundabout is rotating. The theory of special relativity can be applied in Σ. When the observer O' places his measuring rod along a diameter of the roundabout in *Figure 9.4*, the length of the measuring rod is perpendicular to its direction of motion and the rod should not be

Lorentz contracted in the direction of its length relative to Σ. Let O' measure the length of the diameter of the roundabout to be X units. When O' measures the circumference of the roundabout by placing the measuring rod along the periphery of the roundabout,

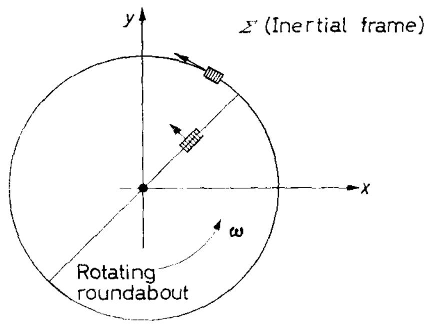

Figure 9.4. A circular roundabout rotates relative to the inertial reference frame Σ. An observer on the rotating roundabout sets out to measure the diameter and circumference of the rotating roundabout. When his measuring rod is placed along the diameter of the roundabout, it is moving perpendicular to its length, and its length is not Lorentz contracted relative to Σ. When placed along the circumference of the rotating roundabout, the measuring rod is moving parallel to its length and should be Lorentz contracted relative to Σ

$O's$ measuring rod is moving in a direction parallel to its length, as shown in *Figure 9.4*, and the measuring rod is Lorentz contracted to

$$l_0\sqrt{(1 - v^2/c^2)} = l_0\sqrt{(1 - r^2\omega^2/c^2)} \simeq l_0(1 - \tfrac{1}{2}r^2\omega^2/c^2)$$

relative to Σ. Hence, relative to Σ, O' must measure the circumference to be greater than $\pi X(=3{\cdot}142X)$ units. Now the measurements of the positions of the ends of $O's$ measuring rod as O' moves along the periphery of the roundabout is a series of events. These events are independent of any particular co-ordinate system. Hence relative to the reference frame in which the roundabout is at rest, the circumference must also be measured to be greater than πX by a factor $\sim(1 - r^2\omega^2/2c^2)^{-1}$.

Since, in the reference frame in which the roundabout is at rest, the circumference of a circle is not equal to π times its diameter, Euclidean geometry is not applicable on the rotating roundabout. Since the length of the measuring rod varies with its radial position,

the ratio of the circumference to the diameter varies with radial position. Furthermore, the rates of clocks fixed to the rotating roundabout depend on their positions, so that it is impossible to synchronize clocks fixed at various positions on the rotating roundabout. In the theory of special relativity we used one co-ordinate system for the whole of space, and using the principle of the constancy of the velocity of light one could, in principle, use light signals to synchronize all the clocks at rest in that co-ordinate system. This is not possible in a rotating co-ordinate system. One must change one's ideas of space and time in the presence of the centrifugal gravitational field. One cannot use one single co-ordinate system for the whole of space, but one must use infinitesimal local co-ordinate systems and allow the metric to vary from point to point.

Relative to the stationary roundabout, the distant stars would have a velocity $r\omega$, and for sufficiently large values or r, the stars would be moving relative to O' with linear velocities exceeding 3×10^8 m/sec (the terrestrial value of the velocity of light). At first sight this appears to be a contradiction of the conclusion derived in Section 5.3.2, that the velocities of all material bodies must be less than c. However, the restriction $u < c = 3 \times 10^8$ m/sec is restricted to the theory of special relativity. According to the general theory, it is possible to choose local reference frames in which, over a limited volume of space, there is no gravitational field, and relative to such a reference frame the velocity of light is equal to c. However, this is not true when gravitational fields are present. In addition to the lengths of rods and the rates of clocks the velocity of light is affected by a gravitational field (cf. Section 9.7). If gravitational fields are present the velocities of either material bodies or of light can assume any numerical value depending on the strength of the gravitational field. If one considers the rotating roundabout as being at rest, the centrifugal gravitational field assumes enormous values at large distances, and it is consistent with the theory of general relativity for the velocities of distant bodies to exceed 3×10^8 m/sec under these conditions.

9.9. REFERENCE FRAMES ACCELERATING RELATIVE TO THE FIXED STARS

It was pointed out in Section 1.5 that, when one is in a train, one can tell when the train is accelerating. If one wants to apply Newton's laws in a reference frame having a linear acceleration relative to an inertial frame Σ, one has to introduce inertial forces.

These were assumed to be fictitious in Section 1.5. If the laws of physics are to be the same in all references frames, then these inertial forces must come naturally out of the full theory of general relativity. It was illustrated in Section 9.2 how it would be impossible for an astronaut in a space-ship to say, without looking at something external to the space-ship, whether the space-ship was undergoing a jerky accelerated motion due to its rockets firing intermittently, or whether large masses were passing near the space-ship giving rise to gravitational fields. Thus one cannot separate the effects due to acceleration from the effects due to a gravitational field. The reader has probably noticed that, when he is in an elevator, the parcels he is carrying are heavier or lighter depending on the acceleration of the elevator.

Relative to an inertial frame the 'fixed' stars are at rest or moving with uniform velocity. However, relative to a reference frame accelerating relative to an inertial frame the stars are accelerating. It is quite feasible that accelerating masses give different gravitational forces from the gravitational forces due to the same masses when they are moving with uniform velocity (cf. electromagnetic case discussed by Rosser[15], Section 9.5). The conditions in an accelerating reference frame are different from the conditions in inertial frames, since the stars are accelerating relative to the accelerating reference frame. It seems plausible to try to interpret inertial forces as gravitational forces due to the accelerations of the stars relative to the reference frame chosen. The laws applicable in the accelerating reference frame would be the general laws, and could be applied in inertial frames also, but in the latter case the laws might take simpler forms. [For example, in the electromagnetic case, the fields $\mathbf{E}$ and $\mathbf{B}$ reduce to $\mathbf{E} = q\mathbf{r}/4\pi\varepsilon_0 r^3$ and $\mathbf{B} = 0$, if $\mathbf{u} = \dot{\mathbf{u}} = 0$ relative to an inertial reference frame.] The events observed are independent of any particular reference frame, but the actual measures of physical quantities depend on the arbitrary standard of rest chosen.

Since the rates of clocks and the lengths of rods are influenced by a gravitational field one cannot use one single reference frame with synchronized clocks distributed throughout space, in the presence of a gravitational field. One can only use infinitesimal local coordinate systems, and must allow the rates of clocks, such as atomic clocks, to vary from point to point depending on the gravitational field, which in turn depends on the distribution of matter.

It should be noted that, by suggesting that inertial forces are due to the gravitational fields of distant stars, we have gone further than the way in which we used the principle of equivalence in

Sections 9.2 to 9.7. There it was merely postulated that measurements in an accelerating reference frame would give the same numerical results as similar experiments performed in a stationary reference frame in which there was a gravitational field of appropriate strength. In Sections 9.8 and 9.9 it was suggested that, relative to non-inertial reference frames, the inertial forces may be interpreted as actual gravitational fields due to the acceleration of the distant masses of the universe relative to the reference frame chosen, that is, inertial forces are gravitational forces.

9.10. THE GENERAL THEORY OF RELATIVITY

It has been illustrated how, in the presence of a gravitational field, one cannot use Euclidean geometry. Einstein[16a] postulated that 'all gaussian co-ordinate systems are essentially equivalent for the formulation of the general laws of nature'.

According to the principle of covariance, the laws of nature should have the same mathematical form in all these gaussian co-ordinate systems. In order to develop a covariant theory, Einstein had to use general tensor analysis. This goes beyond the scope of the present book. A reader interested in the full theory is referred to Einstein[16] and to Born[17] for popular accounts of the theory. For a more comprehensive popular account the reader is referred to Eddington[9]. For a full mathematical account of the theory the reader is referred to Eddington[18] and to the books by Møller, Tolman and Fock quoted previously.

Einstein assumed that due to the effects of gravitational fields on on standard rods and clocks, the distribution of matter determines the local space-time geometry in a particular gaussian co-ordinate system. When there is no gravitational field the space-time geometry is the Minkowski space-time geometry of the theory of special relativity. According to Einstein the paths of light rays in non-Euclidean space-time geometries are geodesics. Using his full theory of general relativity Einstein was able to calculate the gravitational shift of spectral lines and the deflection of light near the sun [which is twice that given by eqn (9.17)]. It has been shown that these results can be obtained to second order v^2/c^2 from the principle of equivalence, without using the full theory of general relativity. Einstein's full theory gave Newton's theory of gravitation as a first approximation, but there were small differences between the two theories. Einstein predicted that the orbit of the planet Mercury is not a 'static' ellipse, but the perihelion of the orbit should precess by 43″ of arc per century. Experimentally, it has

been found that the total precession of the perihelion of the orbit of Mercury is about 5,600″ of arc per century. All but about 40″ of arc per century can be accounted for by the perturbations due to the gravitational attractions of the other planets. Einstein's theory accounts for the extra 40″ of arc per century.

In the general theory the distribution of matter determines the metric of the gaussian co-ordinate system chosen. The equations of the theory of the general relativity are expressed in the form of differential equations. The solutions of these equations depend on the boundary conditions. It is still a matter of some controversy as to how much effect distant stars have on terrestrial phenomena. It was suggested in Section 9.8 that in rotating reference frames, inertial forces can be interpreted as gravitational forces arising from the rotation of distant stars. The best way to determine how the motion of a body affects its gravitational field would be to perform laboratory experiments. However, the gravitational field is very weak and the expected changes in the gravitational field would only be of order v^2/c^2 so that there is no prospect of laboratory measurements to settle the question. There is therefore room for considerable speculation. There are some people who go so far as to claim that all the inertial properties of matter can be attributed to the gravitational attractions of distant stars. For a popular account of this view the reader is referred to Sciama[11]. Sciama has been criticized by Bridgman[19]. For an alternative view of the theory of gravitation the reader is referred to Fock[20].

It is not easy to see what it is that happens to clock A, which undergoes the journey illustrated in *Figure 8.1*, which can make clock A read less than clock B, which remains at rest in an inertial frame (the earth). The difference between the readings of the clocks is generally attributed to the accelerations of clock A relative to an inertial frame. However, the actual magnitude of the difference in the readings of the clocks depends on how long, and at what *uniform* speed, clock A travels relative to the earth. It has been illustrated in this chapter how the distant masses of the universe may affect the inertial properties of terrestrial bodies. According to Mach's Principle the rotation of a body should not be treated in isolation from the rest of the matter in the universe. Similarly, one should probably not consider the two clocks in complete isolation, but one should realize that they have different motions relative to the rest of the matter in the universe, even when the rocket is moving with uniform velocity relative to the earth. Consequently, the clocks may experience different gravitational fields due to the rest of the matter in the universe, leading in turn to different clock

readings. For a discussion of an example of the clock paradox, which takes into account the rest of the matter in the universe, the reader is referred to Whitrow[21].

According to Sciama[11], the inertia of matter is due to gravitational forces due to the distant matter of the universe. When discussing the clock paradox in terms of a travelling twin, Sciama writes[11]:

> Either man may be regarded as stationary; the difference between them springs from their different relations to distant matter. The clock paradox thus has exactly the same status as Newton's experiment with the rotating bucket of water. Some people have claimed that the two men are, in fact, symmetrical, and so should age equally during the journey. They have clearly overlooked the relevance of distant matter. If there were no such matter the men would indeed be symmetrical, and would age equally, but in that case there would be no such thing as inertial frames, or inertia, and no rocket would be needed to accelerate *A*. Life would be quite different in such a universe.

Thus according to Sciama's speculative view, in the absence of the rest of the matter in the universe, that is in a universe consisting of only two clocks, it is possible that Dingle's view might be correct. Our primary concern should, however, be with the actual universe as it is. A substantial amount of evidence has been presented in the last two chapters in favour of the alternative view that, in the actual universe we live in, for the case illustrated in *Figure 8.1*, the travelling twin would age less than the twin who remained on the earth. As long as controversy persists, the most satisfactory solution is to perform as many experiments as possible, even though, in the author's *opinion*, such experiments would almost certainly prove Dingle to be wrong.

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SUPPLEMENTARY READING

For popular accounts of the general theory the reader is referred to *Relativity. The Special and the General Theory* by Einstein, *Einstein's Theory of Relativity* by Born, and *Space, Time and Gravitation* by Eddington.

For an English translation of Einstein's original papers the reader is referred to *The Principle of Relativity* (Dover Publications Inc). For an introduction to the mathematical theory, the reader is referred to *The Einstein Theory of Relativity* by Lieber and Lieber. For mathematical accounts of the general theory the reader is referred to *The Meaning of Relativity* by Einstein, *The Mathematical Theory of Relativity* by Eddington, *The Theory of Relativity* by Møller, *The Theory of Relativity* by Pauli, *The Theory of Space Time and Gravitation* by Fock, *Relativity, Thermodynamics and Cosmology* by Tolman and *An Introduction to the Theory of Relativity* by Bergmann.

Approximate derivations of the gravitational red shift, the deflection of light in the gravitational field of a star and the precession of the perihelion of Mercury are given in *Electromagnetism and Relativity*, by E. P. Ney. For an account of experimental work the reader is referred to Dicke[2].

PROBLEMS

Problem 9.1—Describe how the gravitational shift of spectral lines and the bending of light in a gravitational field can be interpreted in terms of the principle of equivalence.

Problem 9.2—Calculate the fractional difference in frequency between the frequency of an atomic process on top of the spire of Salisbury Cathedral (altitude 404 ft or 124 m) and the frequency of the same process at ground level. (Take g to be 9·81 m/sec^2.)

Problem 9.3—Calculate the difference in the readings of clocks at sea level and at the top of Mount Everest respectively after one year (altitude 29,000 ft or 8842 m). Take the acceleration due to gravity to be a constant equal to 9·81 m/sec.

Problem 9.4—A satellite is in a circular orbit at an altitude of 4,500 km above the earth. Calculate the fractional difference between the frequency of an atomic process in the satellite and the frequency of the same atomic process on the surface of the earth. The radius of the earth is equal to 6367 km.

Problem 9.5—Describe how inertial forces are interpreted in the general theory of relativity.

Problem 9.6—Two identical atoms are hung over a pulley by a weightless thread. Initially, one of the atoms (labelled 1) is at a lower level than the other atom (labelled 2). The lower atom (number 1) is in an excited state, of excitation energy E_0. According to the theory of special relativity the mass of atom 1 is E_0/c^2 greater than atom 2 which is in the ground state. Consequently the heavier atom (number 1) will move vertically downwards pulling atom number 2 upwards. Whilst it is moving downwards, atom 1 emits a photon of energy E_0, which is subsequently absorbed by atom 2. Atom 2 is now heavier than atom 1 and now atom 2 moves vertically downwards pulling atom 1 upwards. Some time after it has passed the level of atom 1, atom 2 emits a photon of energy E_0, which is subsequently absorbed by atom 1 which then moves vertically downwards. If the process is repeated indefinitely work is being done continuously. This is therefore an example of a perpetual motion machine. What is the fallacy in the above argument? [Hint: Remember the gravitational shift of spectral lines. Reference: Frisch[22].]

APPENDIX 1

EXPERIMENTS ON THE MOTION OF THE EARTH RELATIVE TO THE ETHER

(*a*) *Michelson–Morley Experiment*

The most famous of the optical experiments to measure the velocity of the earth relative to the hypothetical ether was the Michelson–Morley experiment[1]. The experiment will be interpreted in terms of the mechanical theories of the ether which were in vogue in the nineteenth century. The reader should treat this Appendix as an exercise in the History of Science. He should attach the same significance to the mechanical ether theories as he does to the caloric theory of heat. If he prefers, the reader can identify the Absolute System, merely with a system in which the speed of light in empty space is the same in all directions. Eqns (A1.1), (A1.2) and (A1.3) then follow from the Galilean velocity transformations, eqns (1.10), (1.11) and (1.12).

The physical principles underlying the Michelson–Morley experiment will be illustrated by using the analogy of a swimming race. Just as the wind carries sound with it, the movement of the ether was supposed to carry light waves with it. Similarly, a river carries a swimmer along with it. The flow of the river affects the speed of the swimmer, whatever the direction the swimmer is going. Let A and B be two points on opposite banks of a river which is flowing at a speed v relative to the ground as shown in *Figure A1.1.* Let l be the width of the stream from the diving board A to the other bank. Let C be a point which is at a distance l upstream from A. We shall consider a race between two swimmers who go from A to B and back and from A to C and back respectively. It will be assumed that each swimmer is able to swim with the same speed c in still water. The question is, which swimmer will win the race? Obviously in still water ($v = 0$) it would be a dead heat, but how will the times of the swimmers compare when there is a current?

In order to allow for the current, the swimmer going from A to B must aim upstream such that his resultant velocity is in the direction A to B. This resultant velocity is equal to $(c^2 - v^2)^{\frac{1}{2}}$, so that the time to go from A to B is equal to $l/(c^2 - v^2)^{\frac{1}{2}}$. Similarly, on the return journey from B to A the swimmer must again aim

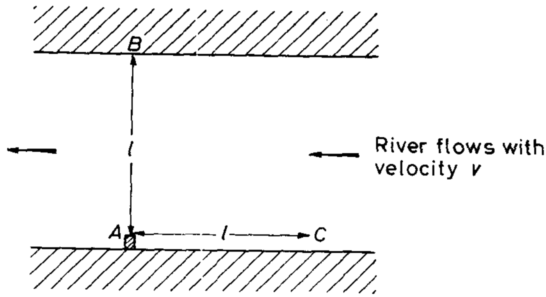

Figure A1.1. A swimming race in which one swimmer swims from A to B and back whilst the other swims from A to C and back. According to the classical ether theories this is equivalent to the Michelson–Morley experiment

upstream and his time is again equal to $l/(c^2 - v^2)^{\frac{1}{2}}$. The total time to go from A to B and back is equal to

$$t_1 = \frac{2l}{c\sqrt{1 - v^2/c^2}} \tag{A1.1}$$

The other swimmer will have a velocity $c - v$ on the journey from A to C and a velocity $c + v$ on the return journey from C to A, so that the total time in this case is given by

$$t_2 = \frac{l}{c - v} + \frac{l}{c + v} = \frac{2lc}{c^2 - v^2} = \frac{2l}{c(1 - v^2/c^2)} = \frac{t_1}{\sqrt{1 - v^2/c^2}} \tag{A1.2}$$

Now v must be less than c otherwise the second swimmer would never reach the point C, thus $1/(1 - v^2/c^2)^{\frac{1}{2}}$ must be greater than unity, so that t_2 is greater than t_1 by a factor $1/(1 - v^2/c^2)^{\frac{1}{2}}$. If one knew c and l and measured the two times t_1 and t_2 with a stop watch, one could estimate v.

An optical experiment which, according to the ether theories, was equivalent to the above swimming race, was carried out by Michelson and Morley in 1887. The principle of their interferometer is illustrated in *Figure A1.2*. Light from a source S is divided at a half-silvered mirror A, part of the light going in the direction AB and part in the direction AC. The light is reflected by mirrors at B and C. The two pencils of light then combine to produce interference fringes, which are viewed by the telescope T. For an account of the theory of the Michelson interferometer the reader is referred to a text book on optics.

For purposes of discussion it will be assumed that there is an 'ether wind' of velocity v in the direction C to A in the laboratory

system as shown in *Figure A1.2. In vacuo*, and for all practical purposes in air, there is no dragging of the ether, so that according to the ether theories the velocity of light relative to the laboratory should be $c - v$ in the direction A to C, $c + v$ in the direction C to A, and $(c^2 - v^2)^{\frac{1}{2}}$ in the directions A to B and B to A. The experiment is therefore equivalent to the swimming race. If $AB = AC = l$ then the times for the light to go from A to B and

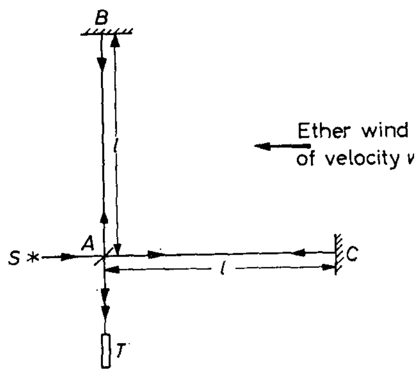

Figure A1.2. Schematic form of the Michelson–Morley experiment

back and A to C and back are given by eqns (A1.1) and (A1.2) respectively.

Hence,

$$\begin{aligned}\Delta t &= t_2 - t_1 \\ &= \frac{2l}{c}\left\{\frac{1}{(1 - v^2/c^2)} - \frac{1}{\sqrt{1 - v^2/c^2}}\right\} \qquad \text{(A1.3)} \\ &= \frac{2l}{c}\{(1 - v^2/c^2)^{-1} - (1 - v^2/c^2)^{-\frac{1}{2}}\}\end{aligned}$$

Expanding by the binomial theorem and neglecting terms of order greater than v^2/c^2,

$$\Delta t = \frac{2l}{c}\{1 + v^2/c^2 - 1 - \tfrac{1}{2}v^2/c^2\} = \frac{lv^2}{c^3}$$

This corresponds to a path difference Δ_1, given by

$$\Delta_1 = c\,\Delta t = \frac{lv^2}{c^2}$$

If the interferometer is rotated through 90 degrees such that the ether wind is from A to B, then the time to go from A to B and back is greater than the time to go from A to C and back by an amount lv^2/c^3, and the path difference is now given by

$$\Delta_2 = -\frac{lv^2}{c^2}$$

Hence, the total change in path difference on rotating the interferometer through 90 degrees is equal to $2lv^2/c^2$. If λ is equal to the wavelength of the light used, then this change in path difference corresponds to a fringe shift of

$$n = \frac{2l}{\lambda}\frac{v^2}{c^2}$$

Michelson and Morley* had a path length of 11 metres, and used a wavelength of $5{\cdot}9 \times 10^{-7}$ m. If it is assumed that the ether wind is equal to the full velocity of the earth in its orbit around the sun, namely 30 km/sec, then $v/c \simeq 10^{-4}$, hence

$$n = \frac{2 \times 11}{5{\cdot}9 \times 10^{-7}} \times (10^{-4})^2 \simeq 0{\cdot}37$$

This is a fraction of a fringe. To ensure accuracy and freedom from vibration, particularly when the interferometer was rotated, Michelson and Morley floated the apparatus on mercury and took their readings continuously as the whole apparatus rotated slowly. The maximum displacement found by Michelson and Morley was less than one hundredth of a fringe. They could have detected a fringe shift of 0·37 fringe quite easily. Now it might have happened that at the time they performed the experiment the earth might just have been at rest relative to the ether. However, when they performed the experiment at different times of the day and also six months later, when the earth was moving in the opposite direction relative to the sun, Michelson and Morley did not find any fringe shift. There have been many repetitions of the Michelson–Morley experiment, but apart from some results by Miller (1925), they all found no fringe shift. In 1958 Cedarholm, Bland, Havens and Townes[2] carried out an extremely accurate experiment on ether drift using two maser beams. Their results showed that the ether drift, if it exists, is less than 1/1000 of the earth's orbital velocity.

How was the null result of the Michelson–Morley experiment interpreted at the time it was performed? Nowadays one tends to

* Michelson and Morley's account of their experiment is reprinted in *Source Book in Physics* by W. F. Magie, published by McGraw-Hill.

lose perspective about the significance of the null result of the Michelson–Morley experiment. It was for this reason that the swimming race analogy was used. The null result of the Michelson–Morley is equivalent to a dead heat in the swimming race. How would the reader try to account for a dead heat? The first suggestion might be that the velocity of the river was zero. Another might be that the swimmers swam at the same speed despite the flow of the river, but this would not be consistent with the Galilean transformations, if the swimmers were able to swim at the same rate relative to still water. One would soon start thinking that the lengths AB and AC were not equal, or that the watches used to time the swimmers were wrong. Similar interpretations were suggested to account for the null result of the Michelson–Morley experiment. One suggestion was that $v = 0$ always, that is, that the reference system in which the ether was assumed to be at rest was always at rest relative to the earth. However, this would give the earth an omnipotent position in the universe which people had been loathe to accept since the time of Copernicus.

Another explanation offered was that the earth pulled the ether in its immediate vicinity around with it, so that the ether near the earth's surface was always at rest relative to the earth, but the earth moved relative to the rest of the ether in the universe. To test the possibility that moving bodies might drag the ether surrounding them with them, Lodge carried out an experiment in which an interferometer was placed between two large steel discs 1 m in diameter. The steel discs were 2·5 cm apart. The steel discs were mounted on a common axle which could be rotated at a very high speed. The interferometer remained at rest during the experiment. If the ether between the plates was dragged by the plates when they rotated, then it was thought that there might be a displacement of the interference fringes when the rotation started. No displacement was observed and it was concluded that the ether was not dragged along by the rotating steel discs.

Another explanation of the Michelson–Morley experiment was suggested by FitzGerald and independently by Lorentz. The suggestion was that bodies moving relative to the ether might be contracted in the direction of motion through the ether such that a length l_0 would be reduced to $l_0(1 - v^2/c^2)^{\frac{1}{2}}$, where v is the velocity of the body relative to the ether. Thus the interferometer arm AC shown in *Figure A1.2* would be reduced to $l_0(1 - v^2/c^2)^{\frac{1}{2}}$; this would make t_1 and t_2 [given by eqns (*A1.1*) and (*A1.2*) respectively] equal to each other. Since rulers would also be contracted in the same proportion, the length of AC would still be measured to be l.

At the time it was suggested, this was an *ad hoc* assumption, but Lorentz was later able to account for such a contraction in a causal way in terms of his electron theory (Rosser[3], Section 2.4). Lord Rayleigh suggested that if a transparent medium did undergo a real contraction due to its motion relative to the ether, then it might become doubly refracting, just as a block of glass does when it is subjected to unilateral stress. Lord Rayleigh[4] attempted to detect the effect experimentally but with negative results. Brace[5] carried out the experiment with increased accuracy but also found no effect, so that motion relative to the hypothetical ether apparently produced no state of strain.

Another possible interpretation of the Michelson–Morley experiment was that the velocity of light was the same in the direction *AC* as in the direction *AB*. Now according to the ether theories this would be equivalent to saying that the swimmers would always swim at the same speed relative to the bank so that the swimming race would be a dead heat, whatever speed the river was flowing. Such an explanation was incompatible with ether theories in which the velocity of light had to be related to a transmitting medium and the Galilean velocity transformations were used. This is, however, precisely one of the assumptions Einstein made in the theory of special relativity. According to the principle of the constancy of the velocity of light, the speed of light in empty space is the same in all directions of space in all inertial reference frames. The acceptance of this postulate necessitated the abandonment of the Galilean transformations and a revision of the concepts of space and time in the way described in Chapter 3. Einstein also postulated that the laws of optics and electromagnetism obeyed the principle of relativity, in which case, there is *no* absolute reference frame for the description of optical and electrical phenomena.

Since the advent of quantum theory it has not been necessary to interpret light as a continuous wave motion in a continuous mechanical medium. It is now believed that light consists of individual photons. These photons are treated as one species of fundamental particle. The wave properties of light are associated with individual photons in a similar statistical way to the way 'wave' properties are associated with individual electrons. The more advanced reader may be interested in trying to interpret the action of the Michelson–Morley interferometer from the quantum mechanical point of view. (Reference: Frisch[6]). (Where does the principle of the constancy of the velocity of light operate in this interpretation?)

(*b*) *Trouton–Noble Experiment*

Maxwell's equations do not obey the principle of relativity, if the co-ordinates and time are changed according to the Galilean transformations. Before the advent of the theory of special relativity, it was believed that Maxwell's equations would be valid in one reference frame only. If so, it should have been possible to identify this absolute system by means of electrical experiments, such as the Trouton–Noble experiment[7].

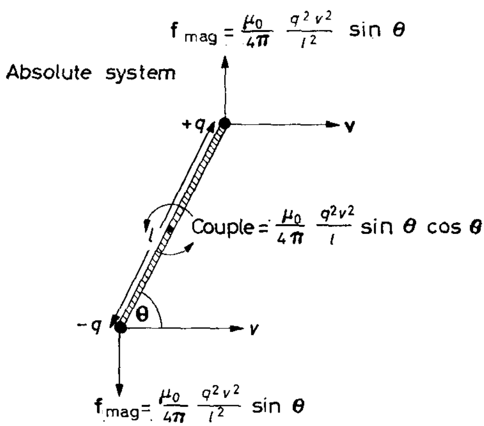

Figure A1.3. The Trouton–Noble experiment. The charges $+q$ and $-q$ are moving with velocity v relative to the absolute system in which it is assumed that Maxwell's equations must be applied. Relative to the absolute co-ordinate system the magnetic fields due to the moving charges give rise to a couple tending to rotate the apparatus in the anti-clockwise direction. No such couple was observed when the experiment was performed

Consider two equal and opposite electric charges $+q$ and $-q$ connected by a 'rigid' rod of length l which is at rest relative to the earth, as shown in *Figure A1.3*. It is unlikely that, if the absolute system exists, the earth is at rest relative to it. Let the velocity of the earth relative to the absolute system be $\mathbf{v}$. If Maxwell's equations must be applied in the absolute system then there is a magnetic force between the charges shown in *Figure A1.3.* Applying the Biot–Savart law one finds that the magnetic force on each charge due to the other charge is equal to

$$f = \frac{\mu_0}{4\pi}\frac{q^2v^2}{l^2}\sin\theta$$

These magnetic forces give rise to a couple of magnitude

$$G = \frac{\mu_0}{4\pi}\frac{q^2v^2}{l^2}\sin\theta\, l\cos\theta$$

If $\mu_0\varepsilon_0 = 1/c^2$ then

$$G = \frac{1}{4\pi\varepsilon_0}\frac{v^2}{c^2}\frac{q^2\sin\theta\cos\theta}{l} \tag{A1.4}$$

This couple tends to rotate the apparatus in the anti-clockwise direction, as shown in *Figure A1.3*, and the apparatus should rotate. The electric force between the charges should be along the line joining them and should not contribute to the couple. By measuring the rotation it should have been possible to estimate v. An experiment of this type was carried out by Trouton and Noble who suspended a charged parallel plate capacitor by a phosphor bronze strip, such that the plates of the capacitor were in the vertical plane. If v were of the order of magnitude of the velocity of the earth in its orbit around the sun, then the couple given by eqn (A1.4), which is of order v^2/c^2, should have been observable in Trouton and Noble's experiment, but the experiment failed to give any indication of such a couple, and it proved impossible to determine any absolute reference frame for electromagnetic phenomena. This suggests that the laws of electromagnetism satisfy the principle of relativity.

It is simplest to interpret the Trouton–Noble experiment from the reference frame in which the two charges shown in *Figure A1.3* are at rest. According to the principle of relativity, the laws of electromagnetism are valid in this reference frame, so that the force between the charges should be given by Coulomb's law for the force between electrostatic charges; there should be no magnetic force between the charges in the inertial frame in which they are both at rest. The resultant force is along the line joining the charges and should not give rise to a couple in the reference frame in which they are at rest, and the charges should not be observed to rotate in any reference frame.

Some of the more advanced readers may like to try and interpret the Trouton–Noble experiment in the reference frame in which the charges are moving with uniform velocity v as shown in *Figure A1.3.* The couple predicted by eqn (A1.4) should still be present. How is it balanced out? Reference: Tolman[8] and Panofsky and Phillips[9].

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APPENDIX 2

MAXWELL'S EQUATIONS AND THE PRINCIPLE OF THE CONSTANCY OF THE VELOCITY OF LIGHT

For the benefit of readers familiar with Maxwell's equations it will now be illustrated how the principle of the constancy of the velocity of light follows, if it is assumed that Maxwell's equations are correct and obey the principle of relativity.

Consider a field point P in empty space at x, y, z and at a time t in the inertial frame Σ as shown in *Figure A2.1(a)*. Let the system of accelerating charges give rise to an electric and magnetic field at P. For a field point in empty space, $\mathbf{B} = \mu_0\mathbf{H}$ and $\mathbf{D} = \varepsilon_0\mathbf{E}$ so that in M.K.S. units, Maxwell's equations take the form

$$\operatorname{curl}\mathbf{E} = -\mu_0\frac{\partial\mathbf{H}}{\partial t} \tag{A2.1}$$

$$\operatorname{curl}\mathbf{H} = \varepsilon_0\frac{\partial\mathbf{E}}{\partial t} \tag{A2.2}$$

$$\operatorname{div}\mathbf{E} = \operatorname{div}\mathbf{H} = 0 \tag{A2.3}$$

where ε_0 and μ_0 are the electric and magnetic space constants in Σ. Taking the curl of eqn (A2.1) and substituting for curl $\mathbf{H}$ from eqn

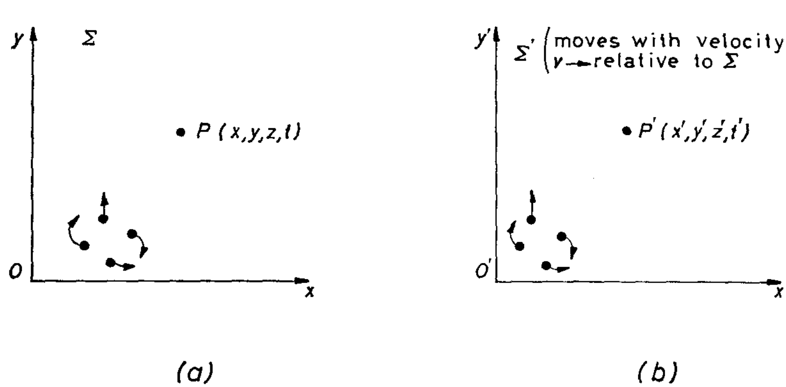

Figure A2.1.(a) The distribution of accelerating charges gives rise to electromagnetic radiation which has a velocity $(\mu_0\varepsilon_0)^{-\frac{1}{2}}$ relative to Σ, at the point P in empty space. (b) The same accelerating charges shown relative to the inertial frame Σ'. The charges give rise to electromagnetic radiation which has a velocity $(\mu_0'\varepsilon_0')^{-\frac{1}{2}}$ relative to Σ' at a point P' in empty space

(A2.2) one obtains

$$\text{curl curl } \mathbf{E} = -\mu_0 \frac{\partial}{\partial t} \text{curl } \mathbf{H} = -\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

Now curl curl $\mathbf{E}$ = grad div $\mathbf{E} - \nabla^2\mathbf{E}$. Since div $\mathbf{E} = 0$ in empty space, one has

$$\nabla^2\mathbf{E} - \mu_0\varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \tag{A2.4}$$

Eqn (A2.4) is a wave equation and is interpreted as describing the behaviour of the electric vector of the electromagnetic radiation at the point P at a time t due to the accelerating system of charges. The velocity of these waves is equal to $(\mu_0\varepsilon_0)^{-\frac{1}{2}}$ and this velocity is identified with the velocity of light. This velocity is independent of the velocity of the charges giving rise to the electromagnetic waves, illustrating how according to Maxwell's equations the velocity of light is independent of the velocity of the source of the light.

Now consider the point P' situated at the point x', y', z' in empty space at a time t' in Σ' as shown in *Figure A2.1(b)*. We shall consider the electric and magnetic fields at P' due to the same system of accelerating charges as was previously considered relative to Σ. In empty space, for the inertial frame Σ', Maxwell's equations become:

$$\text{curl}' \, \mathbf{E}' = -\mu_0' \frac{\partial \mathbf{H}'}{\partial t'} \tag{A2.5}$$

$$\text{curl}' \, \mathbf{H}' = \varepsilon_0' \frac{\partial \mathbf{E}'}{\partial t'} \tag{A2.6}$$

$$\text{div}' \, \mathbf{E}' = \text{div}' \, \mathbf{H}' = 0 \tag{A2.7}$$

where ε_0' and μ_0' are the electric and magnetic space constants relative to Σ'. The co-ordinates in the expressions for curl′ and div′ in eqns (A2.5)–(A2.7) are x', y', z'. For example, div′ $\mathbf{H}'$ is equal to $(\partial H_x'/\partial x') + (\partial H_y'/\partial y') + (\partial H_z'/\partial z')$. The fields $\mathbf{E}'$ and $\mathbf{H}'$ must be defined in the same way relative to Σ', as $\mathbf{E}$ and $\mathbf{H}$ are defined relative to Σ. Taking the curl′ of eqn (A2.5), one obtains the equation

$$\nabla'^2\mathbf{E}' - \mu_0'\varepsilon_0' \frac{\partial^2 \mathbf{E}'}{\partial t'^2} = 0$$

The velocity of electromagnetic waves in Σ' is therefore equal to $(\mu_0'\varepsilon_0')^{-\frac{1}{2}}$. This velocity is again independent of the velocity of the source of the radiation.

The ampere is defined as:

> That unvarying current, which, if present in each of two infinitely thin parallel conductors of infinite length and one metre apart in empty space, causes each conductor to experience a force of exactly 2×10^{-7} newtons per metre of length.

If Maxwell's equations are valid in Σ and Σ', then the force between two parallel currents should be given by the Biot–Savart law in both Σ and Σ'. If the ampere is defined in the same way in Σ and Σ', this would imply that $\mu_0 = \mu_0' = 4\pi \,.\, 10^{-7}$ by definition. For the velocity of light to have the same numerical value in Σ and Σ' one must also have $\varepsilon_0 = \varepsilon_0'$. In principle, according to classical physics, one could place two protons a known distance, say x metres, apart in empty space and measure the force between them when they are at rest in the inertial frame Σ. Similarly one could place two protons x metres apart, and measure the force between them when they are at rest relative to the inertial frame Σ'. If Maxwell's equations are valid in both Σ and Σ', then Coulomb's law should be applicable in both cases. For ε_0 to be equal to ε_0' it would require that the force between the protons at rest in Σ (measured in Σ) would have the same numerical value as the force between the two protons at rest in Σ' (measured in Σ'). Though it is not always stated specifically, it is generally assumed in the theory of special relativity, that, if two experiments are carried out under *identical* conditions in two *inertial* frames Σ and Σ' which are equivalent in every way, then they give the same numerical results, within experimental error. If this assumption is made, then ε_0 should equal ε_0'. Since μ_0 is equal to μ_0' by definition, then $(\mu_0\varepsilon_0)^{-\frac{1}{2}}$ should be equal to $(\mu_0'\varepsilon_0')^{-\frac{1}{2}}$. According to Maxwell's theory these latter expressions are equal to the velocities of electromagnetic waves in free space in Σ and Σ' respectively. Thus, if it is assumed that Maxwell's equations are correct and obey the principle of relativity, then the principle of the constancy of the velocity of light follows. The simultaneous measurement of the velocity of light in Σ and Σ' is not carried out under *identical* conditions in Σ and Σ', since the velocity of the source of light is not the same relative to observers at rest in Σ and Σ' respectively, so that *a priori* there is no reason why they should obtain the same numerical value for the velocity of light; but according to Maxwell's equations they should. With Gaussian units, for the velocity of light to have the same numerical value in Σ and Σ', the ratio of the electromagnetic to the electrostatic units would have to have the same numerical value in Σ and Σ'.

Conversely, if it is assumed that the principle of the constancy of the velocity of light is valid, then this implies that $\varepsilon_0 = \varepsilon_0'$ if Maxwell's equations are correct. Thus, according to the theory of special relativity, the force between two charges at rest in either Σ or Σ' should have the same numerical value under the same conditions in both Σ and Σ'. According to the theory of special relativity Σ and Σ' are completely equivalent for describing magnetic as well as electrostatic phenomena, provided the units are chosen in the same way in Σ and Σ'.

The statement that the velocity of light has the same numerical value in all inertial frames implies that the fundamental units of length and time are defined in the same way in all inertial frames, e.g. as described in Section 1.2.

APPENDIX 3

DERIVATION OF THE LORENTZ TRANSFORMATIONS

(a) Using the Principle of the Constancy of the Speed of Light

The co-ordinates and time of the event of detection of the light at the point P relative to Σ and Σ' in *Figure 3.1* must satisfy the relations.

$$x^2 + y^2 + z^2 - c^2t^2 = 0 \tag{A3.1a}$$

and

$$x'^2 + y'^2 + z'^2 - c^2t'^2 = 0 \tag{A3.1b}$$

The required transformations must transform eqn (A3.1b) into eqn (A3.1a). It is illustrated in Section 3.7, using the principle of the constancy of the velocity of light, that

$$y = y' \tag{A3.2}$$

$$z = z' \tag{A3.3}$$

It will be assumed that the other transformation equations are linear and of the form

$$x' = Ax + Bt \tag{A3.4}$$

$$x = Cx' + Dt' \tag{A3.5}$$

where A, B, C and D are constants independent of x, x', t and t', but the constants A, B, C and D may depend on v, the constant relative velocity of Σ and Σ'.

If the transformation equations were not linear, for example, if

$$x' = Px + Qt^2$$

then putting $x' = 0$ corresponding to the position of the origin of Σ' relative to Σ, we would have

$$x = -(Q/P)t^2$$

in which case the origin of Σ' would be accelerating relative to Σ contrary to our assumption that Σ' and Σ are moving with uniform relative velocity.

Equations (A3.4) and (A3.5) must describe the motions of the origins of Σ and Σ' relative to each other. For example, since the origin of Σ and Σ' coincide at $t = t' = 0$, the position of the origin

of Σ' relative to Σ is given by

$$x = vt \tag{A3.6}$$

Putting $x' = 0$ in eqn (A3.4) corresponding to the position of the origin of Σ', we obtain

$$0 = Ax + Bt, \qquad \text{or} \qquad x = -(B/A)t \tag{A3.7}$$

Comparing eqns (A3.7) and (A3.6),

$$v = -B/A \qquad \text{or} \qquad B = -vA$$

Hence, eqn (A3.4) can be rewritten

$$x' = A(x - vt) \tag{A3.8}$$

If an observer at rest in Σ records the velocity of Σ' relative to Σ as $+v$ along the $+x$ axis, then an observer at rest in Σ' should record the velocity of the origin of Σ relative to Σ' as $-v$ along the x' axis of Σ'. If they did not agree on the magnitude of their relative speed, it could only be due to motion in one direction of space compared with motion in the opposite direction of space. This would contravene the assumption that, relative to any inertial reference frame, all directions in space are equivalent. (Reference: Rosser[1], page 90.) Hence the motion of the origin of Σ relative to Σ', corresponding to $x = 0$ in eqn (A3.5) must be given by

$$x' = -vt' \tag{A3.9}$$

Putting $x = 0$ in eqn (A3.5)

$$0 = Cx' + Dt' \tag{A3.10}$$

Comparing eqns (A3.10) and (A3.9) we conclude that $D = Cv$, so that eqn (A3.5) can be rewritten

$$x = C(x' + vt') \tag{A3.11}$$

Substituting for x' from eqn (A3.8) into eqn (A3.11), one obtains

$$x = C(Ax - Avt + vt')$$

that is

$$t' = A\left[t - \frac{x}{v}\left(1 - \frac{1}{AC}\right)\right] \tag{A3.12}$$

Substitute for x', y', z' and t' from eqns (A3.8), (A3.2), (A3.3) and (A3.12), into eqn (A3.1b), namely

$$x'^2 + y'^2 + z'^2 - c^2t'^2 = 0 \tag{A3.1b}$$

After rearranging,

$$x^2\left[A^2 - \frac{c^2A^2}{v^2}\left(1 - \frac{1}{AC}\right)^2\right] + xt\left[\frac{2c^2A^2}{v}\left(1 - \frac{1}{AC}\right) - 2A^2v\right]$$
$$+ y^2 + z^2 - t^2[c^2A^2 - v^2A^2] = 0 \quad \text{(A3.13)}$$

But x, y, z and t refer to the co-ordinates and time of the event corresponding to the detection of the light at P (cf. *Figure 3.1*). Hence, for the same event, one has, from eqn (A3.1*a*),

$$x^2 + y^2 + z^2 - c^2t^2 = 0 \quad \text{(A3.14)}$$

The coefficients of x^2, xt and t^2 must be the same in eqns (A3.13) and (A3.14). Equating the coefficients of t^2,

$$c^2A^2 - v^2A^2 = c^2$$

or

$$A^2 = \frac{1}{(1 - v^2/c^2)}$$

that is,

$$A = \frac{1}{\sqrt{(1 - v^2/c^2)}} \quad \text{(A3.15)}$$

Equating the coefficients of xt in eqns (A3.13) and (A3.14),

$$\frac{2c^2A^2}{v}\left(1 - \frac{1}{AC}\right) - 2A^2v = 0$$

Since, from eqn (A3.15), A^2 is not equal to zero,

$$\left(1 - \frac{1}{AC}\right) = \frac{v^2}{c^2} \quad \text{(A3.16)}$$

that is,

$$AC = \frac{1}{(1 - v^2/c^2)}$$

Substituting for A from eqn (A3.15), one obtains

$$C = \frac{1}{\sqrt{1 - v^2/c^2}} \quad \text{(A3.17)}$$

Substituting for A and C from eqns (A3.15) and (A3.17) into eqns (A3.8) and (A3.11) respectively,

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}} \quad \text{and} \quad x = \frac{x' + vt'}{\sqrt{1 - v^2/c^2}} \quad \text{(A3.18)}$$

Substituting for A from eqn (A3.15) and for $(1 - 1/AC)$ from eqn (A3.16) into eqn (A3.12), one obtains

$$t' = A\left[t - \frac{x}{v}\left(1 - \frac{1}{AC}\right)\right] = \frac{(t - vx/c^2)}{\sqrt{1 - v^2/c^2}} \qquad \text{(A3.19)}$$

From eqns (A3.18) and (A3.19),

$$t = \frac{(t' + vx'/c^2)}{\sqrt{1 - v^2/c^2}}$$

Collecting the transformations,

$$x' = \frac{(x - vt)}{\sqrt{1 - v^2/c^2}} = \gamma(x - vt)$$

$$y' = y$$

$$z' = z$$

$$t' = \frac{(t - vx/c^2)}{\sqrt{1 - v^2/c^2}} = \gamma(t - vx/c^2)$$

These are the Lorentz transformations. The reader can check that these transformations do transform eqn (A3.1b) into eqn (A3.1a). Collecting the inverse transformations,

$$x = \frac{(x' + vt')}{\sqrt{1 - v^2/c^2}} = \gamma(x' + vt')$$

$$y = y'$$

$$z = z'$$

$$t = \frac{(t' + vx'/c^2)}{\sqrt{1 - v^2/c^2}} = \gamma(t' + vx'/c^2)$$

The Lorentz transformations were derived for a special case only. They can be derived more generally using the theory of linear transformations. (Reference: Stephenson and Kilmister[2].)

(b) Using the Principle of the Constancy of the Limiting Speed of Particles

The experiment of Bertozzi[3] described in Section 5.4.2 showed experimentally that when an electron is accelerated in an electric field, the velocity of the electron does not tend to infinity, as predicted by Newtonian mechanics, but tends to a limiting value of 3×10^8 m/s, namely the speed of light in empty space. If all

inertial reference frames are equivalent, the numerical value of the limiting or ultimate speed should be the same in all inertial frames (provided the units of length and time are the same). If the limiting speed were not the same in all inertial frames, due to the various values of the limiting speed, it would be possible to differentiate between the various inertial frames. This result will be called the principle of the constancy of the limiting speed for particles. It can be used to develop the Lorentz transformations.

Consider an electron travelling at a speed very close to the limiting speed c along the x' axis of Σ'. Let it pass the origin of Σ' at $t' = 0$ and be detected at x' at a later time t' in Σ'. We have $x'/t' \simeq c$. Now the speed of the electron should be greater relative to Σ than Σ', since Σ' is moving relative to Σ along the positive x axis. Thus one expects the speed of the electron to be close to c in Σ also. If the event of detection is measured to be at x at a time t in Σ, we have $x/t \simeq c$. Since Σ' moves relative to Σ, $x' < x$ so that we must have $t' < t$. This illustrates in terms of the motion of a high energy electron that time cannot be absolute.

In *Figure 3.1*, assume that instead of a light pulse, an electron is passing the origins of Σ and Σ' at $t = t' = 0$ and is detected at the point P having co-ordinates x, y, z, t and x', y', z' and t' in Σ and Σ' respectively. Let the speed of the electron be close to the ultimate speed c in Σ'. If its direction of motion is close to the x' axis of Σ', its speed should be greater in Σ than Σ', since Σ' is moving with uniform velocity v relative to Σ along the x axis. Hence, the speed of the electron should be very close to c in Σ. Equations (3.1) and (3.3) are approximately valid for the motion of the electron from the origins of Σ and Σ' to the point P in *Figure 3.1*. One can then use the analysis of Appendix 3(a).

Alternatively, one can develop the expressions for the velocity transformations. For the reasons given in Section 3.4 and Appendix 3(a), assume that the transformations are given by equations (A3.4), (A3.2), (A3.3) and (A3.12) respectively, that is

$$x' = A(x - vt); \quad y' = y; \quad z' = z; \quad t' = A(t - Ex) \qquad \text{(A3.20)}$$

where A and $E = (1 - 1/AC)/v$ are constants.

Proceeding as in Section 4.1,

$$\mathrm{d}x' = A(\mathrm{d}x - v\,\mathrm{d}t); \quad \mathrm{d}t' = A(\mathrm{d}t - E\,\mathrm{d}x)$$

$$u_x' = \frac{\mathrm{d}x'}{\mathrm{d}t'} = \frac{(u_x - v)}{(1 - Eu_x)} \qquad \text{(A3.21)}$$

Now consider an electron moving with a speed very close to the ultimate speed c along the x' axis of Σ', that is $u_x' \simeq c$. The electron

should move with a speed $u_x \simeq c$ in Σ. Substituting in eqn (A3.21) gives

$$E = v/c^2$$

Similarly,

$$u_y' = \frac{dy'}{dt'} = \frac{dy}{A(dt - E\,dx)} = \frac{u_y}{A(1 - vu_x/c^2)} \qquad \text{(A3.22)}$$

Assume that an electron moves in Σ such that $u_x = 0$, $u_y \simeq c$. In Σ' the velocity of the electron should be $u' \simeq c$ having components $u_x' = -v$ and u_y', where, since $u_x = 0$,

$$u_y' = u_y/A \simeq c/A$$

Now,

$$u'^2 = u_x'^2 + u_y'^2$$

Since

$$u' \simeq c \qquad \text{and} \qquad u_y' = c/A,$$

$$c^2 \simeq v^2 + c^2/A^2$$

Hence,

$$A = 1/(1 - v^2/c^2)^{\frac{1}{2}}$$

Substituting for A and E in eqns (A3.20) gives the Lorentz transformations.

REFERENCES

[1] Rosser, W. G. V. *An Introduction to the Theory of Relativity*. 1964. London; Butterworths
[2] Stephenson, G. and Kilmister, G. W. *Special Relativity for Physicists*. Ch. 1., 1958. London; Longmans
[3] Bertozzi, W. *Am. J. Phys.* **32** (1964) 551

APPENDIX 4

SPECIAL RELATIVITY VIA MECHANICS

There is now a large amount of independent experimental evidence in favour of relativistic mechanics, so that one can now abandon the historical approach to special relativity via the principle of the constancy of the speed of light. For example, one can start by describing Bucherer's experimental determination of e/m for fast electrons (cf. Section 5.4.4). It can then be shown how these results can be interpreted, if it is assumed that the mass of a body varies with its velocity $\mathbf{u}$ according to the equation

$$m = m_0/(1 - u^2/c^2)^{\frac{1}{2}}$$

Momentum $\mathbf{p}$ and force $\mathbf{f}$ can then be defined as

$$\mathbf{p} = m\mathbf{u} = m_0\mathbf{u}/(1 - u^2/c^2)^{\frac{1}{2}}$$

and

$$\mathbf{f} = \mathrm{d}\mathbf{p}/\mathrm{d}t$$

The equation of motion of a charge q of rest mass m_0 in an electric field $\mathbf{E}$ and a magnetic field $\mathbf{B}$ can then be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{m_0\mathbf{u}}{\sqrt{1 - u^2/c^2}}\right) = q\mathbf{E} + q\mathbf{u} \times \mathbf{B} \tag{A4.1}$$

As in Section 5.4.5 it can be illustrated how this equation has been used to design proton synchrotrons capable of accelerating protons up to energies of 25 GeV. It should be emphasized that, in addition to the variation of mass with velocity, it is assumed in eqn (A4.1) that q is independent of the velocity of the charge. Equation (A4.1) is only valid if the charge is not radiating. The effects of radiation reaction can be illustrated in terms of the design of high energy electron synchrotrons.

The relativistic mechanics of a single particle can now be developed *precisely* as in Section 5.3 leading up to the expressions

$$T = (m - m_0)c^2$$

$$E = mc^2$$

$$E^2 = p^2c^2 + m_0^2c^4 \tag{A4.2}$$

At this stage one can solve the problem of an electron moving in an electric field (cf. Section 5.4.1) showing how the speed of the electron tends to c. The experimental confirmation of this result is described in Section 5.4.2.

It can be seen from eqn (5.20) that associated with an increase in the kinetic energy of a particle of amount ΔT, there is an increase in its inertial mass equal to $\Delta T/c^2 = \Delta E/c^2$. At this stage in elementary courses, it is generally postulated that the equivalence of mass and energy holds for all forms of energy including the rest mass energies of particles. One can then treat binding energy and nuclear reactions as described in Section 5.8.1. If it is assumed that the laws of conservation of linear momentum and mass-energy are valid; it is then possible, using the formulae developed above, to cover topics such as pair production, meson decay etc. in the way described in Chapter 5. The above theory is adequate for phenomena considered in one reference frame only, for example, the laboratory system. If one wants to use the c.m.-system as well as the laboratory system, one must develop the relativistic transformations. At this stage the Lorentz transformations could be presented as a natural consequence of the extension of the principle of relativity to these new laws of mechanics. In practice, it is simpler to develop the Lorentz transformations from postulates, such as the principle of the constancy of the limiting speed of particles (cf. Appendix 3(b)). Alternatively, one could present the principle of the constancy of the speed of light as a special case of this postulate and proceed as in Chapter 3. However, even at this stage it is not necessary to proceed via the Lorentz transformations. If the new laws of high-speed mechanics obey the principle of relativity, then the rest mass energy of a particle should be the same when the particle is at rest relative to either Σ or Σ', so that m_0c^2 should be an invariant. Similarly, the inertial rest mass of a particle should be the same, whether the particle is at rest relative to either Σ or Σ', so that m_0 and hence c should be invariants. Using eqn (A4.2), we have

$$c^2p^2 - E^2 = c^2p'^2 - E'^2 = -m_0^2c^4$$

The transformations which satisfy these equations are

$$p'_x = \gamma(p_x - vE/c^2); \quad p'_y = p_y; \quad p'_z = p_z; \quad E' = \gamma(E - vp_x)$$
$$\gamma = 1/(1 - v^2/c^2)^{\frac{1}{2}} \tag{A4.3}$$

where v is the velocity of Σ' relative to Σ along their common x axis. These transformations can be used to transform from the centre of mass to the laboratory system and vice versa. They can be applied

to photons to derive the formulae for the Doppler effect in the way described in Section 5.8.3. Since

$$p_x = mu_x; \quad E = mc^2; \quad p_x' = m'u_x'; \quad E' = m'c^2 \text{ etc.,}$$

eqns (A4.3) can be rewritten

$$m'u_x' = \gamma m(u_x - v); \quad m'u_y' = mu_y; \quad m'u_z' = mu_z \tag{A4.4}$$

$$m'c^2 = \gamma mc^2(1 - vu_x/c^2) \tag{A4.5}$$

Dividing eqns (A4.4) by eqn (A4.5) gives the velocity transformations:

$$u_x' = \frac{(u_x - v)}{(1 - vu_x/c^2)}; \quad u_y' = \frac{u_y}{\gamma(1 - vu_x/c^2)}; \quad u_z' = \frac{u_z}{\gamma(1 - vu_x/c^2)} \tag{A4.6}$$

At this stage one can develop the Lorentz transformations from the principle of the constancy of the limiting speed of particles in the way described in Appendix 3(*b*). Alternatively, one can compare eqns (A3.21) and (A3.22) with eqn (A4.6). One can therefore approach special relativity in two ways. In the approach used in the text, special relativity is presented as a consequence of the extension of the principle of relativity to optics and electromagnetism. Alternatively, one can use the approach *outlined* in this Appendix, in which relativistic mechanics is presented as a necessary consequence of *experiments* on high energy particles such as electrons, and the Lorentz transformations as a consequence of the extension of the principle of relativity to these new laws of mechanics.

APPENDIX 5

THE MÖSSBAUER EFFECT

Emission of γ-rays by a Free Nucleus

Consider first γ-ray emission by a *free* nucleus which decays by γ-ray emission from an excited state of excitation energy E_0 to the ground state, as shown in *Figure A5.1(a)*. If linear momentum is conserved in the decay, then the nucleus must recoil with the same momentum as the emitted photon, as illustrated in *Figure A5.1(b)*. Hence, not all the excitation energy can go into the energy of the emitted photon. If E_γ is the energy of the photon emitted, its momentum is E_γ/c so that the recoil momentum of the nucleus is given by

$$p_r = E_\gamma/c \tag{A5.1}$$

The nucleus is so massive that its recoil velocity is generally very much less than the velocity of light, so that the non-relativistic formula can be used for its kinetic energy, namely

$$T_r = p_r^2/2M = \frac{E_\gamma^2}{2Mc^2} \tag{A5.2}$$

where M is the mass and T_r is the kinetic energy of the recoiling nucleus. From the law of conservation of energy

$$E_0 = T_r + E_\gamma \tag{A5.3}$$

or

$$E_\gamma = E_0 - \frac{E_\gamma^2}{2Mc^2} \tag{A5.4}$$

Since $E_\gamma \ll 2\,Mc^2$, E_γ is approximately equal to E_0, so that eqn (A5.4) can be rewritten, to a good approximation, as

$$E_\gamma = E_0\left(1 - \frac{E_0}{2Mc^2}\right) \tag{A5.5}$$

The relativistic formula can be calculated using eqn (5.116) (cf. problem 5.37). The energy of the photon E_γ is less than E_0 by an amount T_r, as illustrated in *Figure A5.3(a)*. For two typical

cases which will be considered later and whose decay schemes are shown in *Figure A5.2(a)* and *(b)* respectively, namely the 129 keV line in ^{191}Ir and the 14·4 keV line in ^{57}Fe, T_r is equal to 0·045 eV and 0·002 eV respectively.

Absorption of γ-rays by a Free Nucleus

Let a γ-ray be incident upon a nucleus of the type illustrated in *Figure A5.1(a)*, when the nucleus is in the ground state, and let

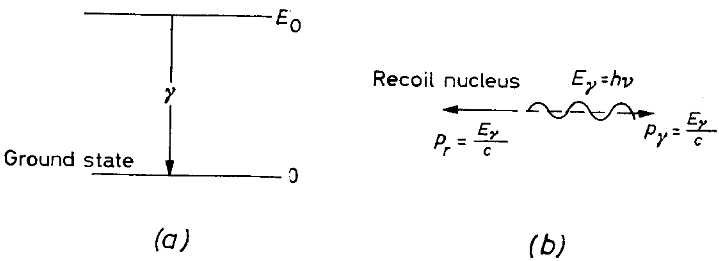

Figure A5.1. The decay of a free nucleus by γ-ray emission

the γ-ray be of the correct energy to excite the nucleus to the excited state of energy E_0 above the ground state. If linear momentum is conserved in the absorption process, the momentum of the recoiling nucleus after absorption will be equal to the momentum of the incident photon. The kinetic energy of recoil is again

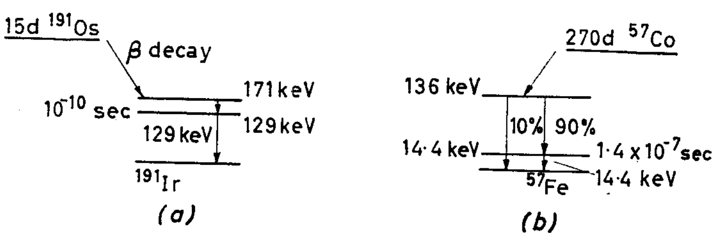

Figure A5.2. The excited states of ^{191}Ir *and* ^{57}Fe

approximately equal to $(E_0^2/2Mc^2) = T_r$, so that, in order to have a high probability of being absorbed, the incident γ-ray must have an energy of $E_0 + T_r \simeq E_0(1 + E_0/2Mc^2)$ as illustrated in *Figure A5.3(a)*. Thus the energy of a photon emitted by a free excited nucleus is normally too low by an amount $2T_r$ to re-excite a similar nucleus from the ground state to the excited state.

Resonance Fluorescence

So far it has been assumed that the excited level E_0 above the ground state always has the same energy. According to the uncertainty principle $\Delta E\, \Delta t \sim \hbar$, so that, if the lifetime of the excited

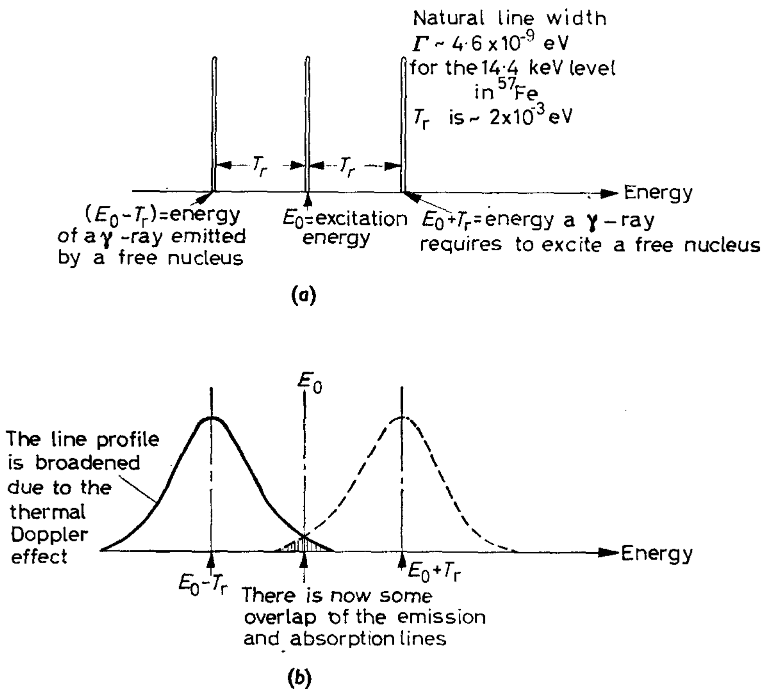

Figure A5.3. In (a), the energy of a γ-ray emitted by a free nucleus, and the energy a γ-ray must have to be absorbed by a free nucleus, are shown. (b) The effect of thermal Doppler broadening is illustrated; the emission and absorption lines may overlap in some cases, so that nuclear resonance fluorescence is possible

state is not infinitely long, it must have a finite energy width. According to quantum mechanics Γ the spread (or uncertainty) in energy is given by

$$\Gamma = \frac{\hbar}{\tau} \tag{A5.6}$$

where τ is the mean lifetime of the excited state, $\hbar$ is equal to $h/2\pi$,

h being Planck's constant. For example, for the 129 keV level in ^{191}Ir and the 14·4 keV level in ^{57}Fe, Γ is equal to $4{\cdot}7 \times 10^{-6}$ eV and $4{\cdot}57 \times 10^{-9}$ eV respectively; these values must be compared with the values of 0·090 eV and 0·004 eV for $2T_r$ for the same levels in ^{191}Ir and ^{57}Fe. The natural line widths are not sufficient to make up the deficit of $2T_r$, so that at first sight it would appear that it would generally be impossible to observe nuclear resonance fluorescence with free atoms unless it were possible, by some method or other, to make up for the deficit of $2T_r$ in the energy of the γ-rays. It was shown in Section 5.8.3 that when a source of light (or γ-rays) is moving relative to the laboratory, then the momenta and energies of the photons in the laboratory are changed according to the momentum and energy transformations of the theory of special relativity. (This is the Doppler effect.) If the nucleus, which emits a γ-ray, is moving relative to the laboratory when it decays, then the energy of the photon relative to the laboratory is changed, being increased if the photon is emitted in the same direction as the direction in which the decaying nucleus is moving. This leads to a spread in the energy of the emitted γ-rays (the Doppler broadening) as illustrated in *Figure A5.3(b)*. In the same way, if the absorbing nucleus is moving, then the absorption line is also Doppler broadened so that the emission and absorption line profiles may sometimes overlap as illustrated in *Figure A5.3(b)*. Moon and his collaborators[1] placed a radioactive source on the tip of a rotator. They found that, if the rotor was rotated at very high speed, the Doppler shift was sometimes enough to make up the gap in energy of $2T_r$, and they observed nuclear resonance fluorescence with the 412 keV level in ^{195}Hg. Since then other techniques have been developed for the study of nuclear resonance fluorescence. If the radioactive source is warmed up sufficiently, the thermal Doppler broadening may be sufficient for nuclear resonance fluorescence to be observed in some cases; this technique sometimes works even at room temperatures. Another method which has been used, is to observe cases when a γ-ray is emitted following β-decay. After emitting a β-ray the nucleus is recoiling, and if the γ-ray is emitted whilst the nucleus is still recoiling, the γ-ray energy is sometimes Doppler broadened sufficiently to give rise to nuclear resonance fluorescence. If the absorbing nucleus is at rest, the probability for nuclear resonance fluorescence depends on the natural line width of the absorbing nucleus, which is related to the mean lifetime by eqn (A5.6). Nuclear resonance fluorescence techniques have been used extensively in the measurement of the lifetimes of the excited states of nuclei. (The reader is referred

to Metzger[2] for a review of the methods used and the results obtained.)

The Mössbauer Effect

If an excited nucleus forms part of the crystal lattice of a solid when it decays, then, under certain conditions, the γ-ray may sometimes be emitted with no momentum and kinetic energy transfer to the nucleus after decay. (For a discussion of the appropriate conditions the reader is referred to Burcham[3] and to Lustig[4]. It will be adequate for our purposes to accept that it can occur.)

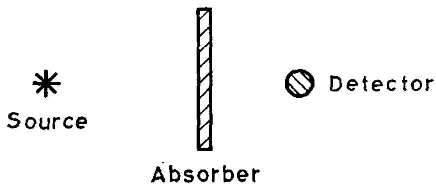

Figure A5.4. An experimental arrangement for observing the Mössbauer effect. For a ^{57}Co *source an iron absorber (generally enriched in* ^{57}Fe) *is used.*

In these cases of recoilless emission, the whole of the crystal effectively takes up the recoil momentum so that practically all the excitation energy goes into the emitted γ-ray in these cases. The energy of the γ-rays is (for most practical purposes) equal to E_0, the excitation energy and the spread in the energy of the γ-rays is due almost entirely to the natural line width arising from the finite lifetime of the excited state. This amounts to $4{\cdot}7 \times 10^{-6}$ eV for the 129 keV level in ^{191}Ir and $4{\cdot}57 \times 10^{-9}$ eV for the 14·4 keV level in ^{57}Fe. Not all the decays are recoilless, but with ^{191}Ir and ^{57}Fe the number of recoilless decays amounts to a few per cent or more.

If the γ-rays are passed through an absorber, which contains the same type of nuclei in the ground state as gave rise to the γ-rays which were emitted without recoil, as shown in *Figure A5.4*, then it is possible to have recoilless absorption of the γ-rays. The recoilless absorption line is also very sharp. Its width is approximately equal to the natural line width. This is much sharper than the thermal Doppler widths necessary for absorption by free atoms namely $4{\cdot}57 \times 10^{-9}$ eV compared with 0·002 eV for ^{57}Fe at 300°K. If the recoilless emission line and the recoilless absorption lines overlap significantly, then there is a strong probability that those γ-rays which were emitted without recoil will be absorbed without

recoil when they pass through the absorber in the experimental arrangement shown in *Figure A5.4*. The nuclear resonance fluorescence, under conditions when the emission and the absorption are both recoilless, is known as the Mössbauer effect, in honour of its discoverer. If the source (or the absorber) in *Figure A5.4* is moved, then the energy of the recoilless emission line (or recoilless absorption line) is shifted due to the Doppler effect, and if the velocity of the source (or absorber) is high enough, the recoilless emission and absorption lines will no longer overlap significantly and the

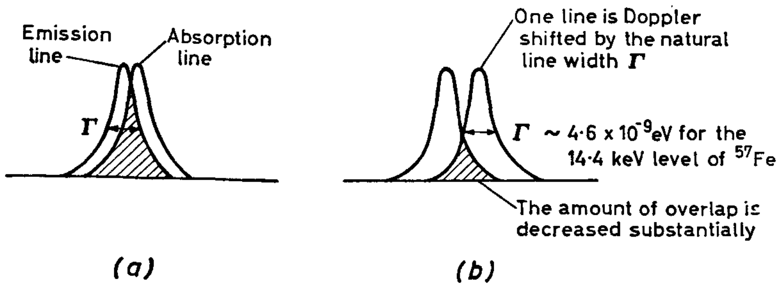

Figure A5.5. The Doppler shift of the emission and absorption recoilless lines reduces the Mössbauer effect

absorption of the γ-rays from the source decreases since the Mössbauer effect is no longer present, so that the detectors in *Figure A5.4* records a higher counting rate. Mössbauer found that for the 129 keV line in ^{191}Ir it was only necessary to move the source at a speed of 1·5 cm/sec to shift the recoilless emission line by its natural line width. This may be contrasted with the enormous speeds required to make up $2T_r$ in the case of γ-ray emission by free nuclei with recoil. The Mössbauer effect absorption cross-section depends critically on the amount of overlap of the recoilless emission and recoilless absorption lines as illustrated in *Figure A5.5*. It can be seen that *small* shifts in the energy (or frequency) of either the recoilless emission or absorption lines changes the probability for the Mössbauer effect substantially and produces measurable effects. Measurements of the Mössbauer effect are an accurate method of measuring differences or changes in frequency. For example with the 14·4 keV line in ^{57}Fe the natural line width is $4{\cdot}57 \times 10^{-9}$ eV and it is possible to measure differences of energy (and frequency) less than the natural line width, so that *changes* in energy (and frequency) of the order of 1 part in 10^{12} can be detected.

The energy of a 14·4 keV γ-ray corresponds to a frequency of $3{\cdot}46 \times 10^{18}$ c/sec. The method has proved accurate enough to be used in the study of the gravitational shift of spectral lines.

In his original experiments, Mössbauer used the 129 keV γ-ray from ^{191}Ir. The recoilless resonant absorption amounted to approximately 1 per cent of the total absorption cross-section at liquid nitrogen temperatures, most of the emission being with recoil. After Mössbauer's discovery a search was made for other substances which could be used, and many more have since been found. The substance which has been used most extensively is ^{57}Fe. A source of ^{57}Co is generally prepared by deuteron bombardment of an iron target (which contains ^{56}Fe) in the form of a strip, the reaction being

$$^{56}\mathrm{Fe} + {}^{2}\mathrm{H} \rightarrow {}^{57}\mathrm{Co} + \mathrm{n}$$

The ^{57}Co is produced near the surface of the iron strip. The ^{57}Co decays by electron capture to excited states of ^{57}Fe and a substantial fraction of the decays go via the 14·4 keV level in ^{57}Fe, which has a mean lifetime of $1{\cdot}4 \times 10^{-7}$ sec, as illustrated in *Figure A5.2(b)*. The lifetime of the 14·4 keV state is long enough for the excited ^{57}Fe ions to occupy suitable sites in the iron crystal lattice before they decay. It is then possible for them to decay without recoil. The 14·4 keV γ-rays can be passed through an iron absorber which can be enriched in ^{57}Fe so as to increase the probability of recoilless absorption. The γ-rays can be detected by a proportional counter or scintillation counter. The probability of recoilless resonance fluorescence is increased in this way and is substantially higher than in ^{191}Ir, and under favourable conditions may amount to 10 per cent or more of the total absorption. With ^{57}Fe the source and absorber can also be used at room temperatures which is a great advantage.

For a fuller review of the Mössbauer effect the interested reader is referred to the articles by Burcham[3] and Lustig[4].

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3 Burcham, W. E. *Sci. Prog., Lond.* **48** (1960) 630
4 Lustig, H. *Am. J. Phys.* **29** (1961) 1

APPENDIX 6

RADAR METHODS AND THE *K*-CALCULUS

(*a*) *Introduction*

Some readers may find it simpler to visualize the determination of the co-ordinates and times of distant events, in terms of the radar methods described in Section 3.9, rather than use all the paraphernalia of imaginary rulers and synchronized clocks distributed throughout space, so that observers, distributed throughout space, can record the co-ordinates and times of events, when and where they happen. Radar methods will be used in this Appendix to develop the Lorentz transformations and to illustrate time dilation and length contraction. The method used is known as the *K*-calculus. The pioneer of this method was Bondi[1,2], whose approach we shall follow. Before proceeding with this Appendix, the reader should familiarize himself with Section 3.9 of Chapter 3.

It will be assumed that there is a radar station at rest on the earth (the laboratory system Σ), which can transmit and receive back radar pulses, which travel with the speed of light. Assume that there is an observer called John in this radar station. Assume that there is also a clock in the radar station on the earth, which can record the times of transmission and receipt of radar signals. It will be assumed that the radar station and observer John are at rest at the origin of the inertial frame Σ.

Consider a rocket moving *directly away* from the earth with uniform velocity v, with a lady astronaut, called Mary, on board. Let the rocket be at rest at the origin of the inertial frame Σ'. Identify the common x axis of Σ and Σ' with the direction of motion of Mary's rocket relative to the earth. It will be assumed that there is a radar set on the rocket, which can transmit and receive radar signals. It will be assumed that Mary has a clock on the rocket, of identical construction to John's clock. [Some readers may prefer to consider both John and Mary in moving rockets, as shown in *Figures 3.4, 3.5* and *3.6* of Sections 3.6 and 3.7. For example, John could be on rocket 1 and Mary on rocket 2.]

Let John send out a radar signal from the earth at a time t_1 on his clock. Let the radar signal be reflected at a distant event on the x axis, and let the reflected radar signal reach the earth at a time t_2 on John's clock. Following the analysis of Section 3.9, if

John assumes that the speed of radar signals (or light) is the same and equal to c in all directions, John will determine the following position (x) and time (t) for the distant event:

$$x = \frac{c}{2}(t_2 - t_1); \qquad t = \tfrac{1}{2}(t_2 + t_1) \tag{A6.1}$$

These equations will be used with different symbols for x, t, t_1 and t_2 throughout this Appendix to measure the co-ordinates and times of distant events relative to John's reference frame, the laboratory system Σ.

Similarly, let Mary send out a radar signal from her rocket at a time t_1' on her clock. Let the signal be reflected from the event on the x' axis, and let the reflected signal reach Mary's rocket at a time t_2' on Mary's clock. Mary will then determine the following position (x') and time (t') for the event:

$$x' = \frac{c}{2}(t_2' - t_1'); \qquad t' = \tfrac{1}{2}(t_2' + t_1') \tag{A6.2}$$

These equations will be used throughout this Appendix to determine the positions and times of events relative to Mary's rocket. Notice the same value used for c in eqns (A6.1) and (A6.2), so as to be in accord with the principle of the constancy of the speed of light which is taken as *axiomatic* in this Appendix. Rather than use radar or light signals, some readers may prefer to use *science fiction* space guns to send messages. These space guns would shoot electrons at such a high speed, that their speed relative to both John and Mary would always be *very*, *very* close to c, and the analysis using radar methods in this Appendix would still be applicable.

(*b*) *Relative Speed of the Earth and Rocket*

Let John send out a series of radar signals from the earth at known times on his clock. Let these signals be reflected by Mary's rocket, and let John determine, on his clock, the times at which he receives these reflected signals. Using eqn (A6.1), John can get a series of values for the positions of Mary's rocket at various times, and hence calculate the speed of Mary's rocket relative to the earth. Similarly, by transmitting radar signals, which are reflected by the earth, Mary can determine the speed of recession of the earth relative to her rocket.

John and Mary must agree on their speed of separation. If they did not obtain the same value for their relative speed, it could only be due to motion in one direction of space compared with motion in the opposite direction of space. This would be contrary to our assumptions that space is isotropic, and that the speed of radar

signals (light) is the same in all directions in all inertial frames. Hence, John and Mary must agree on their speed of separation, which will be denoted by v.

(c) *The K-Calculus*

A space-time diagram, or displacement-time graph will be used to represent the positions and times of events relative to John's reference frame (Σ), which is the laboratory system. As described in Section 3.9, it is conventional to plot $m = ct$ against x, as shown in *Figure A6.1(a)*, where x is the position and t is the time of the event, or world point, measured using John's radar set. The displacement of Mary's rocket relative to the earth is shown in *Figure A6.1(a)*. If Mary and John set both clocks to zero (that is $t = t' = 0$),

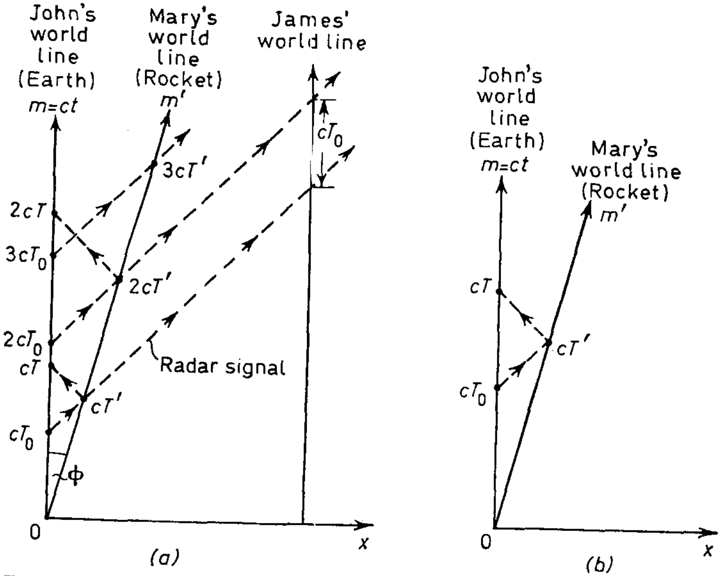

Figure A6.1. (a) John, who is at rest on the earth, transmits radar signals from the earth at the times 0, T_0, $2T_0$, etc. on his clock. The world lines of the signals are shown dotted in Figure A6.1(a). They are at 45 degrees to the x and $m = ct$ axes. Mary is in a rocket moving away from the earth with uniform velocity v relative to John. John's radar signals reach Mary at times 0, T', $2T'$, etc. on Mary's clock. The signals reflected by Mary's rocket return to the earth at times 0, T, $2T$, etc. on John's clock. Later, in Section (j), we shall consider a third inertial observer, James, in a rocket at rest relative to the earth. James receives John's radar signals at time intervals of T_0 on his clock. (b) An enlargement of part of Figure A6.1(a), showing the radar signal John transmits at T_0 on his clock, which is reflected by Mary's rocket at a time T' on Mary's clock, and which returns to earth at a time T on John's clock. It is shown that $T' = KT_0$ and $T = KT' = K^2T_0$, where K equals $(1 + v/c)^{\frac{1}{2}}/(1 - v/c)^{\frac{1}{2}}$

when Mary's rocket passes directly over John's radar station on the earth, then Mary's rocket goes a distance $x = vt = (v/c)ct$ in a time t relative to John. Hence Mary's world line Om' is at an angle $\phi = \tan^{-1}(v/c)$ to the $m = ct$ axis in *Figure A6.1(a)*. Throughout this Appendix it will be assumed that world lines and world points are plotted relative to John's reference frame Σ, using the Ox and Om axes, though we shall take the liberty of labelling the times of events on Mary's world line, Om', as the times measured by Mary's clock.

Let John send out radar signals at times $t = 0$, T_0, $2T_0$ etc., as measured by John's clock at rest on the earth. The world lines of these radar signals are at 45 degrees to the x and $m = ct$ axes in *Figure A6.1(a)*, since in a time Δt light travels a distance $\Delta x = c\ \Delta t$. Let these radar signals be received by Mary's radar set at times 0, T', $2T'$, etc., as measured by Mary's clock on the rocket. These time intervals must be equally spaced, if Mary is leaving the earth with uniform velocity v. Let

$$T' = KT_0 \tag{A6.3}$$

where T_0 is measured on John's clock on the earth, and T' by Mary's clock on the rocket. Now, assume that, whenever Mary receives a signal from the earth, she sends a radio message back to the earth, without time delay, at times 0, T', $2T'$, etc., on Mary's clock. These could be the reflections of John's signals. The world lines of these signals are also shown dotted in *Figure A6.1(a)*. Since the earth is moving away from Mary's rocket with uniform velocity v, the signals reaching John on the earth, should also be equally spaced, say at times 0, T, $2T$, etc. Let

$$T = K'T'$$

where T is measured by John's clock on the earth and T' by Mary's clock on the rocket. According to the postulates of special relativity, all directions in space are equivalent and the speed of light in empty space is the same in all directions in all inertial reference frames, so that there is nothing to differentiate John from Mary. They are just moving apart with uniform velocity v relative to each other in empty space. If Mary measures the interval between John's equally spaced signals to be lengthened by a factor K, when she is going away from John with uniform velocity v, then John should measure the time interval between Mary's equally spaced signals to be increased by the same factor when John is moving away from Mary with the same uniform velocity v. Hence, we

conclude that $K = K'$, so that using eqn (A6.3)

$$T = KT' = K^2T_0 \tag{A6.4}$$

Consider the radar pulse which John transmits from the earth at the time T_0, on his clock, and which is reflected by Mary's rocket at time T' on Mary's clock, and which arrives back on the earth at a time T on John's clock, as illustrated in *Figure A6.1(b)*. Treating the time between $t = 0$ and $t = T_0$ on John's clock on the earth as one interval, we can apply eqns (A6.3) and (A6.4) to these times. Hence, provided Mary's rocket leaves the earth at $t = t' = 0$ and travels directly away from the earth with uniform velocity, so that Mary's world line is a straight line through the origin, as shown in *Figures A6.1(b)* then

$$T' = KT_0 \tag{A6.5}$$

$$T = KT' = K^2T_0 \tag{A6.6}$$

These are the basic formulae of the K-calculus. The quantities T_0, T' and T are illustrated in *Figure A6.1(b)*. *Summarizing*, provided Mary's world line goes through the origin, if John transmits a radar signal at a time T_0 measured by using his clock on the earth, this signal reaches Mary's rocket at a time $T' = KT_0$ on Mary's clock, and after reflection by the rocket this signal reaches the earth at a time $T = K^2T_0$ measured on John's clock. If Mary transmits a radar signal from the rocket at a time T', on her clock, the signal reaches the earth at a time $T = KT'$ measured on John's clock on the earth. The reader should become thoroughly familiar with eqns (A6.5) and A(6.6) and learn to recognize them when they are used with different symbols for T_0, T' and T in this Appendix. The value of the constant K will now be determined.

Consider the radar signal transmitted from the earth at a time T_0 on John's clock and received back after reflection from Mary's rocket at a time T on John's clock. Using eqn (A6.1), John finds the following position for Mary's rocket

$$x = \frac{c}{2}[T - T_0]; \quad t = \tfrac{1}{2}[T + T_0]$$

If Mary's rocket is moving at uniform speed v relative to the earth, since the rocket leaves the origin ($x = 0$) at the time $t = 0$,

$$\frac{v}{c} = \frac{x}{ct} = \frac{T - T_0}{T + T_0}$$

Using eqns (A6.5) and (A6.6),

$$\frac{v}{c} = \frac{K^2 T_0 - T_0}{K^2 T_0 + T_0} = \frac{K^2 - 1}{K^2 + 1} \tag{A6.7}$$

Rearranging, and taking the square root, we find

$$K = \sqrt{\frac{1 + v/c}{1 - v/c}} \tag{A6.8}$$

This determines the factor K. It is illustrated in Section (j) of this Appendix that eqn (A6.5) is an example of the Doppler effect.

Two important expressions involving K will now be calculated

$$K + \frac{1}{K} = \sqrt{\frac{1 + v/c}{1 - v/c}} + \sqrt{\frac{1 - v/c}{1 + v/c}}$$

$$= \frac{(1 + v/c) + (1 - v/c)}{\sqrt{1 - v^2/c^2}}$$

Hence,

$$K + \frac{1}{K} = \frac{K^2 + 1}{K} = \frac{2}{\sqrt{1 - v^2/c^2}} = 2\gamma \tag{A6.9}$$

Similarly,

$$K - \frac{1}{K} = \frac{K^2 - 1}{K} = \frac{2v/c}{\sqrt{1 - v^2/c^2}} = \frac{2\gamma v}{c} \tag{A6.10}$$

where

$$\gamma = (1 - v^2/c^2)^{-\frac{1}{2}} \tag{A6.11}$$

(d) The Lorentz Transformations

It will be assumed that John is at the origin of Σ, that Mary's rocket is at the origin of Σ', and that Mary's rocket passes John's radar station on the earth at $t = t' = 0$. Let an event occur at a point on the x axis, at a point further away from the earth than Mary's rocket. Let John transmit a radar signal at a time $(t - x/c)$, measured on his clock. Let this signal be reflected at the event, and be received back on the earth at a time $(t + x/c)$ on John's clock as illustrated graphically in *Figure A6.2(a)*. Using eqns (A6.1), John estimates the position of the event as x at a time t. Let the radar pulse pass Mary's rocket at a time $(t' - x'/c)$ on Mary's clock, and let the signal reflected by the event pass Mary at a time $(t' + x'/c)$ on Mary's clock, so that, using eqn (A6.2), Mary calculates the position of the event as x' at a time t'.

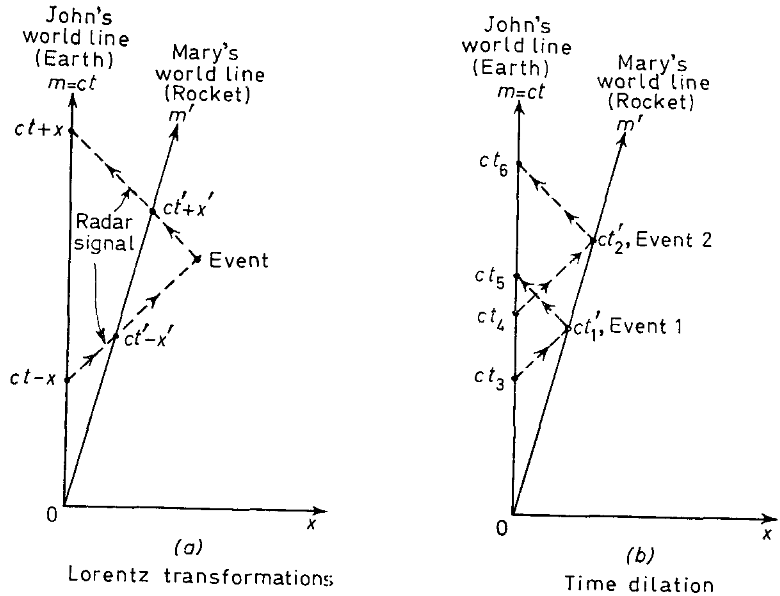

Figure A6.2. (*a*) *Lorentz transformations. John transmits a radar signal, shown dotted in Figure A6.2*(*a*), *at a time* $(t - x/c)$ *on his clock. After reflection at a distant event the signal returns to the earth at a time* $(t + x/c)$ *on John's clock. The radar signal passes Mary at times* $(t' - x'/c)$ *and* $(t' + x'/c)$ *on Mary's clock.* (*b*) *Time dilation. Two events occur on Mary's rocket at times* t_1' *and* t_2', *measured on Mary's clock. John transmits radar signals, at times* t_3 *and* t_4 *on his clock which, after reflection at events* 1 *and* 2 *respectively, return to the earth at times* t_5 *and* t_6 *respectively, measured on John's clock*

Treating the time from 0 to $(t - x/c)$ on John's clock as T_0 and the time from 0 to $(t' - x'/c)$ on Mary's clock as T', we have from eqn (A6.5),

$$ct' - x' = K(ct - x) = Kct - Kx \tag{A6.12}$$

Treating the interval between 0 and $(t' + x'/c)$ on Mary's clock as T' and the interval between 0 and $(t + x/c)$ on John's clock as T, from eqn (A6.6), we have

$$ct + x = K(ct' + x')$$

or

$$ct' + x' = \frac{ct}{K} + \frac{x}{K} \tag{A6.13}$$

Subtracting eqn (A6.12) from eqn (A6.13) gives

$$2x' = x\left(\frac{1}{K} + K\right) - ct\left(K - \frac{1}{K}\right)$$

Substituting for $(K + 1/K)$ and $(K - 1/K)$ using eqns (A6.9) and (A6.10) respectively, gives

$$x' = \gamma(x - vt) \tag{A6.14}$$

where

$$\gamma = (1 - v^2/c^2)^{-\frac{1}{2}} \tag{A6.11}$$

Adding eqns (A6.12) and (A6.13) gives

$$2ct' = ct\left(\frac{1}{K} + K\right) - x\left(K - \frac{1}{K}\right)$$

Using eqns (A6.9) and (A6.10), we obtain

$$t' = \gamma(t - vx/c^2) \tag{A6.15}$$

Equations (A6.14) and (A6.15) are the Lorentz transformations, where x and x' are the positions and t and t' are the times of *the* event measured by John and Mary respectively using radar methods. When using eqns (A6.1) and (A6.2), it was assumed that the speed of light was equal to c in all directions of empty space, relative to *both* John and Mary. This is the *principle of the constancy of the speed of light*, so that the present derivation of the Lorentz transformation is based on precisely the same axioms as the development in Section 3.4. In the present case, it is easier to visualize how the co-ordinates and time of the distant event are determined. Solving eqns (A6.14) and (A6.15) for x and t gives the inverse transformations,

$$x = \gamma(x' + vt'); \quad t = \gamma(t' + vx'/c^2)$$

So far only the measurement of events in the direction of relative motion has been discussed. The example used in Sections 3.6 and 3.7 of Chapter 3, and illustrated in *Figure 3.6*, is in the spirit of the approach adopted in this Appendix, since light signals were used to send information from the events in *Figure 3.6* to two observers C and C' in uniform relative motion. For example, one could put John in rocket 1 and identify him with C, and put Mary in rocket 2 and identify her with C' in *Figure 3.6*, and then use the argument of Section 3.7 to show that

$$y = y' \tag{A6.16}$$

$$z = z' \tag{A6.17}$$

This completes the Lorentz transformations. After developing the Lorentz transformations, we could now use them to develop the expressions for time dilation, length contraction etc., in the algebraic way described in Section 3.5. For the benefit of readers, who

prefer to use the geometric methods of the K-calculus, the expressions for time dilation and length contraction will now be developed, *without* using the Lorentz transformations. The principle of the constancy of the speed of light will, however, be taken as axiomatic.

(*e*) *Time Dilation*

Let two events 1 and 2 happen on Mary's rocket at times t_1' and t_2' respectively, as measured by Mary's clock on the rocket. Let John be fortunate enough to send a radar signal from the earth at a time t_3, on his clock, such that it reflected from Mary's rocket at the time of the first event on her rocket, and let this reflected signal reach John at a time t_5 on John's clock, as illustrated graphically in *Figure A6.2(b)*. Let John send a second radar pulse at a time t_4 on his clock, such that it is reflected from Mary's rocket at the time of event 2, and returns to John's radar station on the earth at a time t_6 on John's clock as illustrated graphically on *Figure A6.2(b)*. Mary estimates the time interval between the events as

$$\Delta t' = t_2' - t_1'$$

Using equations similar to eqns (A6.1), John estimates the time between the two events as

$$\Delta t = \tfrac{1}{2}(t_6 + t_4) - \tfrac{1}{2}(t_5 + t_3) \tag{A6.18}$$

Using eqns (A6.5) and (A6.6), we have

$$t_1' = Kt_3 = t_5/K$$
$$t_2' = Kt_4 = t_6/K$$

Substituting in eqn (A6.18) for t_3, t_4, t_5 and t_6 gives

$$\Delta t = \frac{1}{2}\left(Kt_2' + \frac{t_2'}{K}\right) - \frac{1}{2}\left(Kt_1' + \frac{t_1'}{K}\right)$$
$$= \frac{1}{2}\left(K + \frac{1}{K}\right)\Delta t'$$

Using eqn (A6.9) gives

$$\Delta t = \frac{\Delta t'}{\sqrt{1 - v^2/c^2}} \tag{A6.19}$$

This is an example of time dilation (cf. Sections 3.5.2 and 3.10). The time interval measured by Mary's clock is a proper time interval, since both events happen on the rocket.

(*f*) *Length Contraction*

Consider the measurement of the length of Mary's rocket by both John and Mary. The world lines of the ends of Mary's rocket are illustrated in *Figure A6.3(a)*. Let John transmit a radar signal at a time t_1, on his clock, such that it is reflected from the front of Mary's

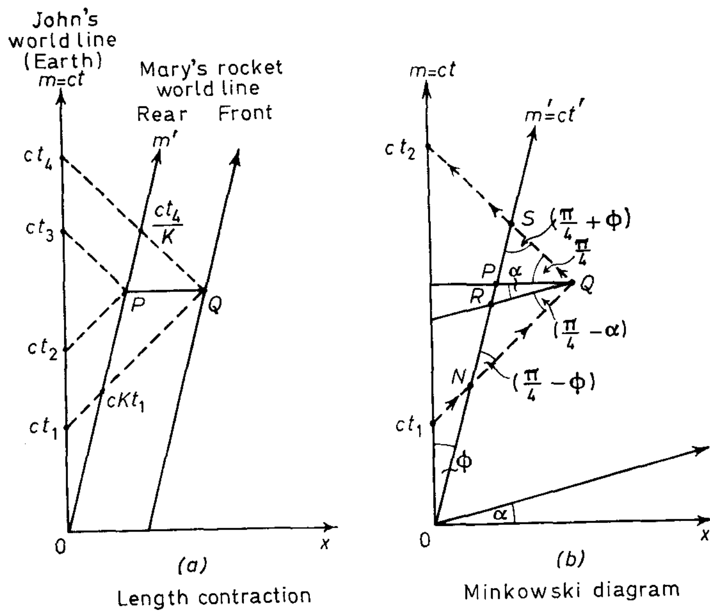

Figure A6.3. (a) Length contraction. The world lines of the front and rear of Mary's rocket are shown. John transmits radar signals at times t_1 and t_2 on his clock. These signals are reflected at the front and rear of Mary's rocket at world points Q and P respectively. Using radar methods, John calculates P and Q to be simultaneous. (b) Minkowski diagram. It is suggested that Ox' and Om' could be used as oblique axes to represent the co-ordinates and times of events relative to Mary's rocket

rocket at the world-point Q in *Figure A6.3(a)*, and returns to the earth at a time t_4, on John's clock, as illustrated graphically in *Figure A6.3(a)*. John estimates the position of the front of Mary's rocket as

$$x_Q = \frac{c}{2}(t_4 - t_1); \quad t_Q = \tfrac{1}{2}(t_4 + t_1) \tag{A6.20}$$

Let John transmit a second radar signal at a time t_2 on John's clock, such that it is reflected at the rear of Mary's rocket at the world-point P in *Figure A6.3(a)* and returns to the earth at a time t_3 on

John's clock, as illustrated in *Figure A6.3(a)*. John estimates the position of the rear of Mary's rocket as

$$x_P = \frac{c}{2}(t_3 - t_2); \quad t_P = \tfrac{1}{2}(t_3 + t_2) \tag{A6.21}$$

Now, in order to measure the length of the rocket, John must measure the positions of the front and rear of Mary's rocket at the same time, as measured by John's radar experiments. Hence, t_Q must equal t_P at the world points P and Q, as shown in *Figure A6.3(a)*. Hence, we must have

$$t_4 + t_1 = t_3 + t_2 \tag{A6.22}$$

Subtracting eqn (A6.21) from eqn (A6.20) and using eqn (A6.22), John concludes that the length of the moving rocket is

$$l = x_Q - x_P = c(t_4 - t_1)/2 - c(t_3 - t_2)/2$$

$$l = c(t_4 - t_3) = c(t_2 - t_1) \tag{A6.23}$$

Let Mary stand at the rear of her rocket. According to eqn (A6.5), the radar signal, transmitted by John at a time t_1 on his clock passes the rear of Mary's rocket at a time Kt_1 on Mary's clock. According to eqn (A6.6), after reflection from the front of Mary's rocket at the world point Q in *Figure A6.3(a)*, this radar signal passes Mary at the rear of her rocket at a time t_4/K on her clock. Using eqn (A6.2), Mary therefore concludes that the distance from the rear to the front of her rocket is

$$l_0 = \frac{c}{2}\left(\frac{t_4}{K} - Kt_1\right)$$

Rearranging,

$$t_4 = \frac{2Kl_0}{c} + K^2 t_1 \tag{A6.24}$$

From eqn (A6.6), we obtain in the present case,

$$t_3 = K^2 t_2 \tag{A6.25}$$

Subtracting eqn (A6.25) from eqn (A6.24) gives

$$t_4 - t_3 = \frac{2l_0 K}{c} - K^2(t_2 - t_1)$$

From eqn (A6.23), both $(t_4 - t_3)$ and $(t_2 - t_1)$ are equal to l/c. Hence,

$$l = 2Kl_0 - K^2 l$$

Rearranging, and using eqn (A6.9), gives

$$l = \frac{2Kl_0}{1 + K^2} = l_0\sqrt{1 - v^2/c^2} \tag{A6.26}$$

This is the expression for the Lorentz contraction. John measures the length of Mary's rocket to be less than Mary does.

(*g*) *Simultaneity of Spatially Separated Events*

Consider the two events of John's measuring the extremities of Mary's rocket at the world points P and Q in *Figure A6.3(a)*. Relative to John's reference frame Σ, these events are simultaneous. Mary records the event at P at a time t'_P, where according to eqns (A6.5) and (A6.6),

$$t'_P = Kt_2 = \frac{t_3}{K} = \frac{1}{2}\left(Kt_2 + \frac{t_3}{K}\right) \tag{A6.27}$$

According to eqn (A6.5), the signal which John transmitted at the time t_1 on his clock, passes the rear of Mary's rocket at a time Kt_1 on Mary's clock, whereas according to eqn (A6.6), the signal, reflected from the front of Mary's rocket at the world point Q, passes the rear of Mary's rocket at a time t_4/K on Mary's clock. Mary concludes that the time of the event at the world point Q, is

$$t'_Q = \frac{1}{2}\left(Kt_1 + \frac{t_4}{K}\right) \tag{A6.28}$$

Since from eqn (A6.22), $(t_2 - t_1)$ equals $(t_4 - t_3)$, subtracting eqn (A6.28) from eqn (A6.27) gives

$$\begin{aligned} t'_P - t'_Q &= \tfrac{1}{2}K(t_2 - t_1) + \tfrac{1}{2}(t_3 - t_4)/K \\ &= \frac{1}{2}\left(K - \frac{1}{K}\right)(t_2 - t_1) \end{aligned} \tag{A6.29}$$

Subtracting eqn (A6.21) from eqn (A6.20) and using eqn (A6.22) gives

$$\begin{aligned} x_Q - x_P &= c(t_4 - t_1)/2 - c(t_3 - t_2)/2 \\ &= c(t_4 - t_3)/2 + c(t_2 - t_1)/2 = c(t_2 - t_1) \end{aligned}$$

Substituting $(x_Q - x_P)/c$ for $t_2 - t_1$ and for $\left(K - \dfrac{1}{K}\right)$ from eqn (A6.10) into eqn (A6.29) gives,

$$t'_P - t'_Q = \gamma v(x_Q - x_P)/c^2$$

This is in agreement with the Lorentz transformations, (cf. Section 3.6). Thus, according to eqns (A6.27) and (A6.28), Mary does not

record the events at P and Q as simultaneous. Thus simultaneity is not an absolute property, as it was assumed to be in Newtonian mechanics. Mary concludes that John did not measure the ends of her rocket at the same time on her reckoning, (cf. Sections 3.5.1, 3.6 and 3.7 of Chapter 3). The reader should remember that we took the principle of the constancy of the speed of light as axiomatic, when applying eqns (A6.1) and (A6.2).

(*h*) *Minkowski Diagrams*

In *Figures A6.1, A6.2* and *A6.3,* the space-time or displacement-time diagrams always represented the co-ordinates and times of events relative to the earth, that is John's reference frame Σ. Now the world line of Mary's rocket corresponds to the motion of the origin of Σ' relative to Σ, that is the line $x' = 0$. Hence, it might be possible to use the line Om' in *Figure A6.3*(b) as a time axis to represent the times of events relative to Mary's reference frame Σ'.

Consider an event at the world point Q in *Figure A6.3*(b). Let John transmit a radar signal at a time t_1 on his clock, which after passing Mary's rocket at the world point N, in *Figure A6.3*(b) is reflected at the event at the world point Q and after passing Mary's rocket at the world point S reaches the earth at the time t_2 on John's clock. If the line Om' were used as a time axis for Mary's reference frame, the time, corresponding to the event at Q in *Figure A6.3*(b), must be at the point R on the Om' axis, which is half-way between the points S and N, if Mary uses eqn (A6.2) to calculate the time of the event at Q. Thus events along the line RQ in *Figure A6.3*(b) must be measured to be simultaneous relative to Mary's rocket, if Om' is used as Mary's time axis. The line corresponding to $t' = 0$, should be a line through the origin O parallel to RQ. Hence it might be possible to use Ox', which corresponds to $t' = O$, as the x' axis to represent the co-ordinates of events relative to Mary's rocket (Σ'). If the world line of Mary's rocket, that is the line Om', makes an angle ϕ to the $m = ct$ axis in *Figure A6.3*(b), then, since radar signals are at an angle of $\pi/4$ relative to both the x and $m = ct$ axis in *Figure A6.3*(b), the angles RNQ and RSQ are $(\pi/4 - \phi)$ and $(\pi/4 + \phi)$ respectively. If the angle PQR is α, then the angles SQR and RQN are $(\pi/4 + \alpha)$ and $(\pi/4 - \alpha)$ respectively. Hence in the triangle QRN,

$$\frac{NR}{RQ} = \frac{\sin\left(\dfrac{\pi}{4} - \alpha\right)}{\sin\left(\dfrac{\pi}{4} - \phi\right)} \tag{A6.30}$$

In the triangle RQS

$$\frac{RS}{RQ} = \frac{\sin\left(\frac{\pi}{4} + \alpha\right)}{\sin\left(\frac{\pi}{4} + \phi\right)} \tag{A6.31}$$

Since $NR = RS$, equating the right-hand sides of eqns (A6.30) and (A6.31) shows that $\alpha = \phi$. Hence RQ, and hence Ox', are inclined at the same angle ϕ to the Ox axis as the Om' axis is inclined to the Om axis in *Figure A6.3(b)*. Thus Ox' and Om' can be used as oblique axes to represent the co-ordinates of events, relative to Mary's rocket (Σ'). It is shown in Chapter 6, that, if we choose the lengths of the units of x' and $m' = ct'$ appropriately, that is using eqns (6.5) and (6.6) of Chapter 6, then one can use the *same* world points and the *same* world lines relative to Mary's rocket (Σ'), that is the oblique Ox', Om' axes, as relative to John's reference frame Σ, that is the rectangular Ox, Om axes. The world points such as Q and world lines are then invariants. For example, the position of the point Q need not be changed relative to the Ox', Om' axes. The more adventurous reader may prefer to proceed to Chapter 6 for a full discussion of Minkowski diagrams in Section 6.2, and may like to develop the transformations of the theory of special relativity using the more elegant 4-vector methods in Section 6.4, before returning to consider the applications of these transformations in Chapters 3, 4 and 5.

(*i*) *Velocity Transformations*

Consider a rocket moving along the x axis, away from both John and Mary, with astronaut Peter abroad. Let Peter's speed relative to Mary be w, as measured by Mary using radar methods. Let Peter's speed relative to John, who is at rest on the earth, be u as measured by John using radar methods. Let

$$K_1 = \left(\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}\right)^{\frac{1}{2}}; \quad K_2 = \left(\frac{1 + \frac{w}{c}}{1 - \frac{w}{c}}\right)^{\frac{1}{2}}; \quad K_3 = \left(\frac{1 + \frac{u}{c}}{1 - \frac{u}{c}}\right)^{\frac{1}{2}}$$

If John transmits radar signals at intervals T_0 on his clock, according to eqn (A6.5), they are received at time intervals of $K_1 T_0$ by Mary, as measured by her clock, and time intervals of $K_3 T_0$ by Peter, as measured by his clock. Now, if Mary transmits signals at intervals of $K_1 T_0$, on her clock, whenever she receives one from John,

according to eqn (A6.5), Peter should receive Mary's signals at intervals of $K_2K_1T_0$. However, the interval measured by Peter should still be equal to K_3T_0, just as if Mary had not received signals and re-transmitted them without time delay. Hence,

$$K_3T_0 = K_2K_1T_0, \quad \text{or} \quad K_3^2 = K_2^2K_1^2$$

$$\left(\frac{1+\frac{u}{c}}{1-\frac{u}{c}}\right) = \left(\frac{1+\frac{w}{c}}{1-\frac{w}{c}}\right)\left(\frac{1+\frac{v}{c}}{1-\frac{v}{c}}\right)$$

Solving for u gives

$$u = \left(\frac{w+v}{1+\frac{vw}{c^2}}\right)$$

This equation is consistent with the velocity transformations developed in Section 4.1.

(j) The Doppler Effect

Consider again the example illustrated in *Figure A6.1(a)*. Let ν_0 $(= 1/T_0)$ be the frequency of signals transmitted by John from the earth, as measured by John's clock. Let ν' $(= 1/T')$ be the frequency of the signals received by Mary on the moving rocket, as measured by Mary's clock. Since by definition, that is eqn (A6.3), T' equals KT, we have

$$\nu' = \frac{\nu_0}{K} \tag{A6.32}$$

or,

$$\nu' = \nu_0\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}} \tag{A6.33}$$

One can treat each successive maximum of a plane monochromatic wave as a signal. Hence eqn (A6.33) should hold for light. Equation (A6.33) is the relativistic expression for the Doppler effect, for the case when the source and observer are moving directly away from each other with uniform relative velocity v. (Compare Sections 3.10.2, 4.4.4 and 5.8.3.) When the source is going directly away from the observer the frequency of the light goes down.

Let astronaut James be in a rocket which is at *rest* relative to the earth. Let him be further away from the earth than Mary's rocket. James' world is shown in *Figure A6.1(a)*. It is parallel to the $m = ct$ axis. Consider the radar signals, which John transmits at times 0, T_0, $2T_0$ etc., on John's clock. Since James is at rest relative to John, the time intervals between the successive signals, received by James must be equal to T_0. Let Mary transmit signals, whenever she receives one from John at times 0, T', $2T'$ on her clock, where $T' = KT_0$ etc. These signals should also reach James at intervals of T_0, just as if there had been no intermediate stage, when Mary received and re-transmitted radar signals. Hence, if Mary transmits signals of frequency ν_0' $(= 1/T')$ measured by her clock, the frequency of signals reaching James must be $\nu = (1/T_0)$, when Mary is approaching James. Thus if the source is approaching the observer directly with uniform velocity v,

$$\nu = K\nu_0' = \nu_0' \sqrt{\frac{1 + \dfrac{v}{c}}{1 - \dfrac{v}{c}}} \tag{A6.34}$$

In this case the frequency goes up by a factor K. Notice, if the frequency of light from a receding source goes down from ν_0 to ν_0/K, the frequency of light from the same light source, when the source is approaching the observer with the same uniform velocity goes up from ν_0 to $K\nu_0$. This fact is used in the *gedanken experimente* in Section 8.2.

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ANSWERS TO PROBLEMS

3.3 In Σ, $L = L_0 \left\{\dfrac{\cos^2 \theta'}{\gamma^2} + \sin^2 \theta'\right\}^{\frac{1}{2}}$; $\tan \theta = \gamma \tan \theta'$.

3.4 $1/(2 \times 10^8)$ of its diameter or $\sim 6\cdot37$ cm.

3.5 $0\cdot866\, c$.

3.8 (*a*) The rear of the rocket is 3×10^{10} m from the earth when the light reaches it, 100 sec after the time of emission; (*b*) $0\cdot9\, c$.

3.10 (*a*) 1·36 km; (*b*) 4·6 km; (*c*) 14·5 km.

3.11 68.

3.13 $v = 0\cdot98\, c$.

3.15 (*a*) $\lambda = -573$ Å; (*b*) $-2{,}536$ Å.

3.16 (*a*) $0\cdot8\, c$; (*b*) $0\cdot89\, c$; (*c*) $0\cdot65\, c$.

3.17 (*a*) $v = 0\cdot8\, c$; (*b*) 3 sec.

3.19 $v = c\left\{1 + \dfrac{X^2}{c^2 T^2}\right\}^{-\frac{1}{2}}$

4.1 (*a*) $u_x = 45(1 - 5 \times 10^{-15})$ m/sec; (*b*) $u_x = 30$ m/sec, $u_y = 15(1 - 5 \times 10^{-15})$ m/sec.

4.2 (*a*) $u_x = 0\cdot83\, c$, $u_y = 0$; (*b*) $u_x = 0\cdot1\, c$, $u_y = 0\cdot796\, c$; (*c*) $u_x = 0$, $u_y = 0\cdot798\, c$; (*d*) 97°11′ to the direction of motion.

4.3 $u_{\parallel} = 0\cdot82\, c$, $u_{\perp} = 0\cdot395\, c$.

4.5 For the first case:

$$u^2 = \frac{u'^2 + v^2 + 2u'v \cos \alpha' - (u'v \sin \alpha')^2/c^2}{[1 + (u'v/c^2)\cos \alpha']^2}$$

$$\text{and } \tan \alpha = \frac{\sqrt{(1 - v^2/c^2)}\, u' \sin \alpha'}{u' \cos \alpha' + v}$$

The expression for u is symmetrical in v and u', but $\tan \alpha$ is not.

4.6 (*a*) $0\cdot87\, c$; (*b*) 40 stone; (*c*) She would not appear contracted, but merely rotated relative to the laboratory. The colour of her clothes, hair and complexion would change (cf. Section 4.4.5).

4.7 (*a*) $1\cdot6\, c$; (*b*) $0\cdot8\, c$; (*c*) $0\cdot976\, c$.

4.13 0·69.

4.14 $\Delta\lambda = +6\cdot56$ Å.

4.15 10·6 m.

4.16 75° 30′.

4.18 (*a*) $4\cdot24 \times 10^{18}$, $3\cdot76 \times 10^{18}$, $3\cdot39 \times 10^{18}$, $3\cdot08 \times 10^{18}$, $2\cdot73 \times 10^{18}$ c/sec; (*b*) 1·0: 0·79: 0·64: 0·53: 0·44.

4.20 (*a*) $\Delta\lambda = 1{,}333$ Å; (*b*) $\Delta\lambda = \sim +840$ Å; (*c*) $\sim 53\frac{3}{4}°$.
5.3 $\beta(= v/c)$ is (*a*) 0·196; (*b*) 0·549; (*c*) 0·94.
m/m_0 is (*a*) 1·02; (*b*) 1·197; (*c*) 2·97.
5.4 $\beta(= v/c)$ is (*a*) 0·145; (*b*) 0·42; (*c*) 0·87.
5.6 (*a*) 0·0536 cm; (*b*) 0·227 cm; (*c*) 1·19 cm; (*d*) 3·80 cm.
5.7 31·5 cm.
5.12 (*a*) 36; (*b*) $\sim 2 \times 10^{-18}$.
5.13 (*a*) 34 days; (*b*) $8\frac{1}{4}$ h; (*c*) 300 sec.
5.14 (*a*) 0·079 MeV; (*b*) 0·261 MeV. $m/m_0 =$ (*a*) 1·15; (*b*) 1·51.
5.15 (*a*) $2·56 \times 10^5$ V; (*b*) 0·75 c.
5.16 $1·4 \times 10^{-12}$.
5.20 28 MeV.
5.21 (*a*) 17·3 MeV; (*b*) $-1·1$ MeV.
5.22 (*a*) $4·2 \times 10^9$ kg/sec; (*b*) $4·5 \times 10^{-6}$ N/m².
5.24 144 MeV/c^2 or 282 electron masses.
5.26 $m_1 = 277\ m_e$, $m_2 = 1840\ m_e$; (*a*) $\sim 2{,}380\ m_e$; (*b*) ~ 500 MeV/c; (*c*) ~ 97 MeV.
5.27 $E_{\max} = 52·8$ MeV.
5.32 $u_{\text{c.m.}} = 0·138\ c$; $1·0095\tau$.
5.33 129 MeV/c; 228 MeV.
5.35 0·511 MeV.
5.36 (*a*) 145 MeV; (*b*) 171 MeV; (*c*) 5·6 GeV.
5.40 $2·4 \times 10^{-10}$ sec.
6.5 (*a*) $-c^2$; (*b*) $-m_0^2c^2$.
6.12 $x = \dfrac{c^2}{f}\left[\left\{1 + \dfrac{f^2t^2}{c^2}\right\}^{\frac{1}{2}} - 1\right]$.
8.4 (*a*) $4c/5$; (*b*) 10 years; (*c*) $t_{\text{LAB}} = 20$ years, $t_{\text{ROCKET}} = 12$ years.
8.5 $v = 0·995\ c$.
8.6 $4·8 \times 10^{-5}$ sec.
8.7 (*a*) 670; (*b*) 18·3.
8.8 (*a*) $x = 0·43$ light years, $\tau = 0·875$ years; (*b*) $x = 9$ light years, $\tau = 2·94$ years; (*c*) $x = 99$ light years, $\tau = 5·2$ years.
8.9 (*a*) 18 light years, $\tau = 11\frac{3}{4}$ years; (*b*) 1998 light years, $\tau = 29\frac{7}{12}$ years.
9.2 $\Delta\nu/\nu = 1·35 \times 10^{-14}$.
9.3 30·5 μ sec.
9.4 $\Delta\nu/\nu = 8·4 \times 10^{-11}$.

A TABLE OF PHYSICAL CONSTANTS

This is a short list of those constants necessary to solve the problems set in the text. For a fuller list the reader is referred to E. R. Cohen, J. W. M. DuMond, T. W. Layton and J. S. Rollet *Rev. Mod. Phys.* **27** (1955) 363.

Speed of light

$$c = 2{\cdot}997930 \times 10^{8} \text{ metre . sec}^{-1}$$

Electronic charge

$$e = 1{\cdot}60206 \times 10^{-19} \text{ Coulomb}$$

Electron rest mass

$$m = 9{\cdot}1083 \times 10^{-31} \text{ kg} = 0{\cdot}510976 \text{ MeV}/c^2$$

Charge to mass ratio of the electron

$$e/m = 1{\cdot}75890 \times 10^{11} \text{ Coulomb . kg}^{-1}$$

Proton rest mass

$$M_p = 1{\cdot}67239 \times 10^{-27} \text{ kg} = 938{\cdot}211 \text{ MeV}/c^2$$

Neutron rest mass

$$M_n = 1{\cdot}67470 \times 10^{-27} \text{ kg} = 939{\cdot}505 \text{ MeV}/c^2$$

Ratio proton mass to electron mass

$$M_p/m = 1836{\cdot}12$$

Atomic mass unit (^{16}O scale)

$$1 \ a.m.u. = 1{\cdot}6598 \times 10^{-27} \text{ kg}$$

Atomic mass unit (^{12}C scale)

$$1 \ a.m.u. = 1{\cdot}6603 \times 10^{-27} \text{ kg}$$

Avogadro's number (^{16}O scale)

$$N = 6{\cdot}02486 \times 10^{26} \text{ (kg mole)}^{-1}$$

Planck's constant

$$h = 6{\cdot}62517 \times 10^{-34} \text{ Joule . sec}$$

Boltzmann constant

$$k = 1{\cdot}38044 \times 10^{-23} \text{ Joule . (°K)}^{-1}$$

Electric space constant

$$\epsilon_0 = 8{\cdot}85434 \times 10^{-12} \text{ Farads . metre}^{-1}$$

Magnetic space constant

$$\mu_0 = 4\pi \times 10^{-7} \text{ Henrys . metre}^{-1}$$

Electron volt

$$1 \text{ eV} = 1{\cdot}60206 \times 10^{-19} \text{ Joule}$$

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